A Theory of Quantum Space-time

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Abstract:

An interpretation and re-formulation of modern physics which removes the presumption of the space-time continuum, and bases physical theory on a small number of rational and empirical principles. In so doing it removes paradox, eliminates wave particle duality, and restores the notion of reality independent of observation. After briefly describing the philosophical principles underlying the theory, we rigorously construct a discrete model of quantum mechanics. Special relativity is developed through the *k*-calculus with no presumption of a manifold. Position is a relationship between particles which necessarily contains uncertainty. Kets are labels which categorise states of matter but do not directly describe them. The principle of superposition is shown as a definitional truism in the categorisation of states. This resolves the measurement problem of quantum mechanics and related paradoxes such as Schrödinger's cat by attributing the collapse of the wave function to information. The probability interpretation has a natural meaning in which the configuration of interacting particles plays the role of a hidden variable. The model supports a form of relativistic quantum field theory which does not depend on quantisation or second quantisation from classical mechanics. Continuous laws of wave mechanics are found in a discrete metaphysic which does not involve waves. Classical law is the expected behaviour of many elementary particles. The constraints on the theory are sufficient to establish that particles are point-like entities with specific properties found in nature. Newton's first law and conservation of momentum are established from the principle of homogeneity. Maxwell's equations are derived from the simple interaction in which a Dirac particle emits or absorbs a photon. Feynman rules are calculated for the discrete theory and differ from the standard rules by the removal of the ultraviolet divergence and the use of proper loop integrals. They give finite and unambiguous results.

Keywords:
Quantum Electrodynamics, Foundations of Physics, Elementary Particle, Relativity

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1 Introduction

Some seventy years after Heisenberg's formulation of the uncertainty principle [1] there is continuing discussion regarding its interpretation [2]. At the same time there is increasing interest in the idea that fundamental variables such as time should actually be discrete [3], and there are many references in the literature on the potential quantisation of gravity which suggest that at a fundamental level space-time may be discrete [4]. This paper replaces the assumption of a pre-existent space-time continuum with the observation that time and distance are numbers produced by a measuring apparatus and interprets quantum logic [5] as a mathematical structure which arises naturally from the categorisation of physical states which are not directly measured.

This paper shows that the removal of the space-time continuum makes possible an intuitive explanation of the principle of superposition (i.e the properties of vector space), the probability interpretation and the wave function, as incorporated in the Birkhoff and Von Neumann axioms of quantum mechanics [6], and resolves the measurement problem of quantum mechanics and the related paradoxes, e.g. [6][7], by attributing the collapse of the wave function to information, which alters the categorisation of states without modifying physical reality. While the notion of reality independent of observation is preserved, the interpretation describes the epistemological nature of the laws of quantum mechanics and supports Kant's thesis that knowledge of the world is in part constituted by the mind.

In this paper we develop the theory as far as the derivation of Maxwell's equations and the resolution of the divergence problems of qed. Subsequent papers will show that the force of gravity follows from the definition of the metric by means of discrete interactions, and will describe a form of interaction which agrees with quark confinement and the strong force.

2 Intuitive Law

Although many modern philosophers, following Popper, deny the possibility of a theory of scientific truth, most scientists seem to have some sort of intuitive idea of truth. To clarify what is involved in such an intuitive idea principles $\mathbb{A}1$ - $\mathbb{A}4$ will be used as, for want of a better term, philosophical axioms of science. These are not axioms in the mathematical sense, in that, on their own, they do not provide a basis for logical deduction. Nor are they considered to be obvious, self evident truths; thousands of years of debate shows that they are not. But they define a clear position in this debate, such that the counter position appears unreasonable or anti-scientific, and they enable us to eliminate many theories, and establish a basis for empirical principles with which to induce and abstract further physical law.

$\mathbb{A}1$. *Matter exists*

$\mathbb{A}2$. *The behaviour of matter is reflected in our perceptions* (psycho-physical parallelism).

$\mathbb{A}3$. *A valid description of matter must be free from self contradiction.*

$\mathbb{A}4$. *There are no physical infinities*

A detailed discussion of these axioms is not the purpose of this paper, and here we merely comment that $\mathbb{A}1$ is necessary to avoid solipsism (or similar), $\mathbb{A}2$ is necessary if we are to have any chance of analysing the universe, $\mathbb{A}3$ is necessary if the universe is in any sense understandable, and $\mathbb{A}4$ is necessary to avoid intuitively daft statements, such as that the universe is twice as big as itself, and is often used by physicists, for example to justify the use of integral formulae without analytical proof of convergence. There has been a huge literature since Zeno on the problem of infinity, and we only remark that knowledge of the existence of various infinities in mathematics does not allow us to conclude that any of them exist physically. Although it is not obvious that there is no physical infinity, it appears to us that the alternative is the patently false assertion that infinite processes described in the axioms of set theory can be carried out in deed as well as in thought.

Although the assumption of the physical existence of $\mathbb{R}^n$ is in conflict with $\mathbb{A}4$, infinity is not excluded from theory, because $\mathbb{A}3$ permits mathematical modelling. If we can embed a description of the universe into a mathematical structure which is proven free from contradiction, we can conclude that statements of physics which are logically true within the structure must also be physically true. Such a mathematical structure may well be infinite or contain infinities, such as those implicit in $\mathbb{R}^n$.

Nothing is derived or deduced directly from $\mathbb{A}2$, but it allows induction and abstraction of further laws, $\mathbb{L}1$ - $\mathbb{L}7$, by observation and analysis of physical processes. Induction is clearly dangerous, and must be used carefully. We do not permit arbitrary interpolation or extrapolation of data, which might be inaccurate, incomplete, or cause conflict of principle with $\mathbb{A}4$. To minimise the possibility of conflict it is desirable to identify something approaching a minimum number of laws consistent with $\mathbb{A}1$ - $\mathbb{A}4$ and incorporating sufficient knowledge of the relationships found in matter to specify a mathematical model.

$\mathbb{L}1$. *Matter is composed solely of elementary particles in a finite number of types.*

Reason: Everything can either be subdivided or not. It follows from the prohibition of infinity the process of subdividing matter cannot be carried on indefinitely, and that there must therefore be a smallest piece, an elementary particle.

As we subdivide matter, it is obvious that there is less structure in the matter contained in the pieces. An elementary particle must therefore be the simplest type of physical quantity, and we seek to describe it without preconceptions as to its meaning. In particular we must avoid the notion that particles exist within a co-ordinate space, or a manifold. It is perhaps helpful to think of a finite set of elements, with certain subsets representing different particle types. Into the "set of particles" we introduce the relation, "interacts with". The ability to interact is the only property directly described and attributed to particles, and all observable properties of particles arise from interactions. Any other properties will be deduced.

$\mathbb{L}2$. *Elementary particles interact.*

Reason: They could not otherwise create the structures of matter we observe.

The assumption in $\mathbb{A}3$ is that interactions between particles can be described in mathematics. We can then restrict attention to those interactions whose deduced theoretical properties correspond to the observed behaviour of matter. The ability to interact requires (at minimum) that we postulate a primitive notion of discrete time applied to each particle. Thus each particle has a time-line of discrete instants, such that at each instant the particle has the possibility of an interaction. This introduces a topological notion of connections in matter, such that each instant is "adjacent to" the next instant in the time-line of the particle, and such that when particles interact we identify the instants in their time-lines at which the interaction takes place. This topological notion of position (as distinct from co-ordinate position in a geometric space) can be seen diagrammatically in Figure 6, and is implicit in $\mathbb{L}3$.

$\mathbb{L}3$. *Fundamental physical laws are the same throughout the universe (the principle of homogeneity).*

Reason: The type of interaction available to each particle can only be a property of that particle. Apart from the existence from a finite number of particle types, and the specific configurations of particles, there is nothing in the theory to distinguish any matter in the universe from any other matter in the universe. If the perceived properties of time, space and motion are simply relationships generated by interactions, then these properties are always the same wherever these interactions takes place. After the definition of reference frames (below) it will follow that there is invariance under translation, rotation, inversion of space co-ordinates (not spin) and motion.

3 Co-ordinate Systems

There is room for confusion between two very similar questions, ‘What is time?’ and ‘What is the time?’. The first question has something to do with consciousness, and our perception of time as a flow from past to future. It admits no easy answer, but it is quite distinct from the second question and only the second question is relevant in the definition of space-time co-ordinates. The answer to the question ‘What is the time?’ is always something like 4:30 or 6:25. The time is simply a number read from a clock.

$\mathbb{L}4$. *The universe contains processes (clocks) which can be analysed and used to define a quantity known as the time.*

Reason: These processes are observed so $\mathbb{A}2$ states that they exist.

There are many different types of clock, but every clock has two common elements, a repeating process and a counter. The rest of the mechanism converts the number of repetitions to conventional units of time. A good clock should provide accurate measurement and it should give a uniform measure of time. We cannot count less than one repetition of the process in the clock, so for accurate measurement the process must repeat as rapidly as possible. In a uniform clock, the repeating process must repeat each time identical to the last, uninfluenced by external matter. One repetition gives the minimum unit of time for any given clock. Subdividing this unit of time requires a second clock. So time takes integer values. In principle there may be clocks, i.e. repeating processes, which are faster than any process used in a practical clock, but, by $\mathbb{A}4$, there must be some indivisible process, which determines a smallest notional unit of time, the chronon, called after its name in antiquity. There may be more than one such indivisible repeating process, so the chronon need not be unique. We assume that is very much smaller than the unit given by any practical clock, and that for practical purposes conventional measures of time can be regarded as (large) whole numbers of chronons.

Definition: Let $\chi \in \mathbb{N}$ be the scaling factor to chronons from conventional units of time.

A clock defines the time, but does so only at one place. A space-time co-ordinate system also requires a definition of distance, and a definition of time at a distance from the clock.

$\mathbb{L}5$. *The universe contains a process (radar), which can be analysed and used to define space-time co-ordinates.*

Reason: As Bondi pointed out *“with our modern outlook and modern technology the Michelson-Morley experiment is a mere tautology”* [9]. $\mathbb{L}5$ is true because it depends only on abstraction from observation and tautological definition, not on induction. It tautologically defines space-time co-ordinates only at points where radar is actually used.

Definition: The distance of an event is half the lapsed time for radar to go out and return; the time at which the signal is reflected is the mean time between when it is sent and when it returns.

Radar defines distance in units of time, so space-time co-ordinates are strictly elements of $\mathbb{N}^4$. Radar is preferred to a ruler, because it applies directly to both large and small distances, and because a single measurement can be used for both time and space co-ordinate. Radar also measures direction and it will be seen that the algebra is formally identical for 3-vectors with a Euclidean metric and for one dimensional space-time diagrams, such as figure 1. Each point on a space-time diagram represents an event.

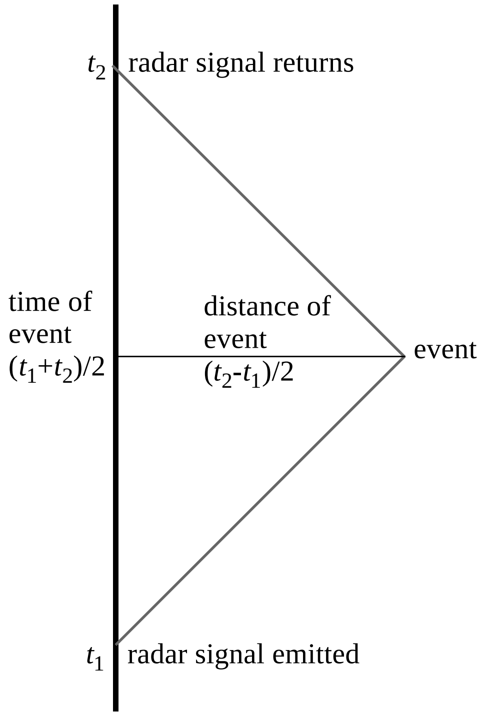


Figure 1: Space-time diagrams are defined such that lines of equal time are horizontal and lines of equal distance are vertical. By definition, uniform motion in the reference frame is shown by a straight line on the diagram. To use radar we must know the speed of light (if distance were defined using a ruler, then to measure the time at an event we would still need to know the speed of a message from the event). But now we have a paradox. To measure speed we conduct a time trial over a measured distance, but first time must be defined at both ends of the ruler, which requires knowledge of the speed of light. We know no other way to measure the time of an event at a distance from a clock; if we synchronise two clocks by bringing them together, we have no guarantee that they remain synchronised when they are separated, unless light is used to test their synchronisation. Thus the speed of light is an absolute constant because measurement of speed requires a co-ordinate system, which requires light for its definition. An experiment to determine the speed of light actually measures the conversion factor from natural units in which the speed of light is 1. Thus, by definition, light is drawn at 45^{o}.

Definition: A space time co-ordinate system defined by radar is known as a reference frame.

A reference frame is a mathematical construction, namely the set of all values which can result from process of measurement, not a physical entity. It depends on the possibility of measurement and cannot be extended indefinitely into space or defined in a perfect vacuum where there is no matter.

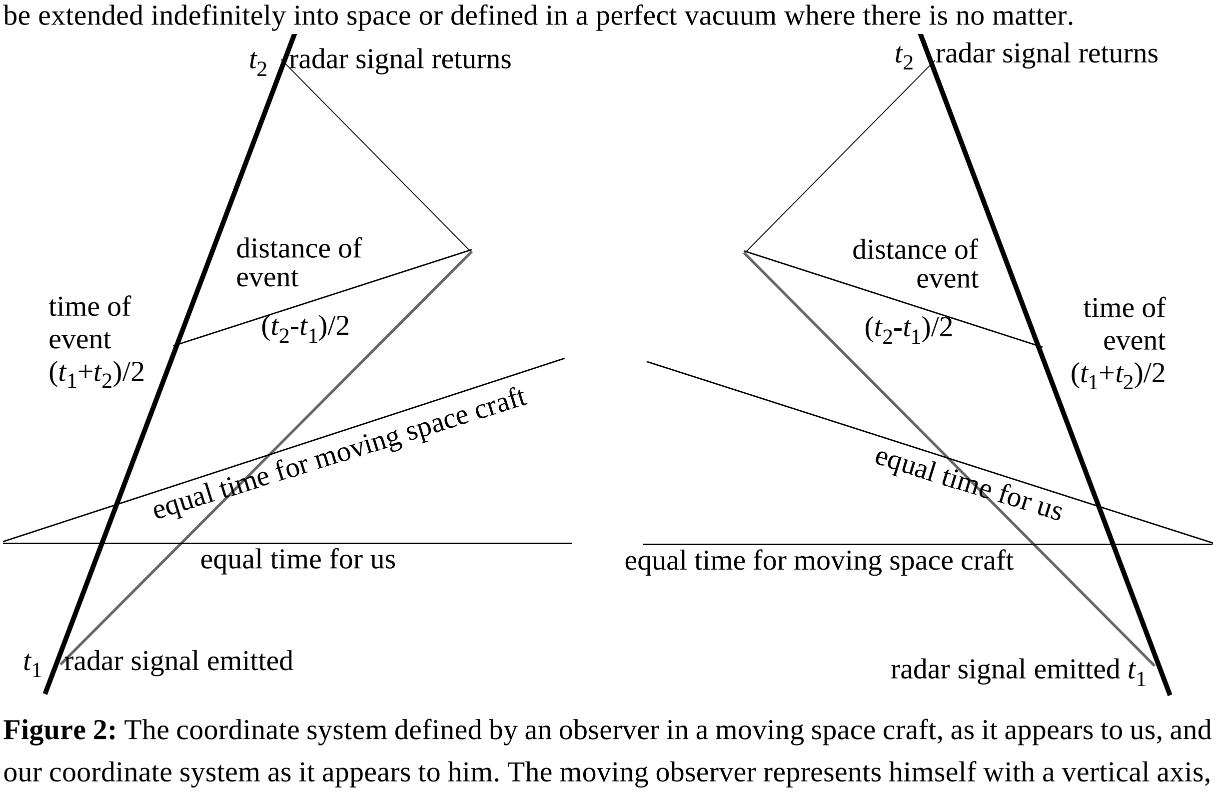


Figure 2: The coordinate system defined by an observer in a moving space craft, as it appears to us, and our coordinate system as it appears to him. The moving observer represents himself with a vertical axis, and he would draw us at an angle. In his diagram our reference frame appears distorted. By $\mathbb{L}3$ the co-

ordinate system of the moving observer is just as valid as our own. Switching from one co-ordinate system to another is Lorentz transformation. To transform co-ordinates, we need to know what unit of time the other scientist is using. There is no way to synchronise the clocks directly, but, according to $\mathbb{L}3$, the principle of homogeneity, two clocks will give the same unit of time if the physical processes in each are identical.

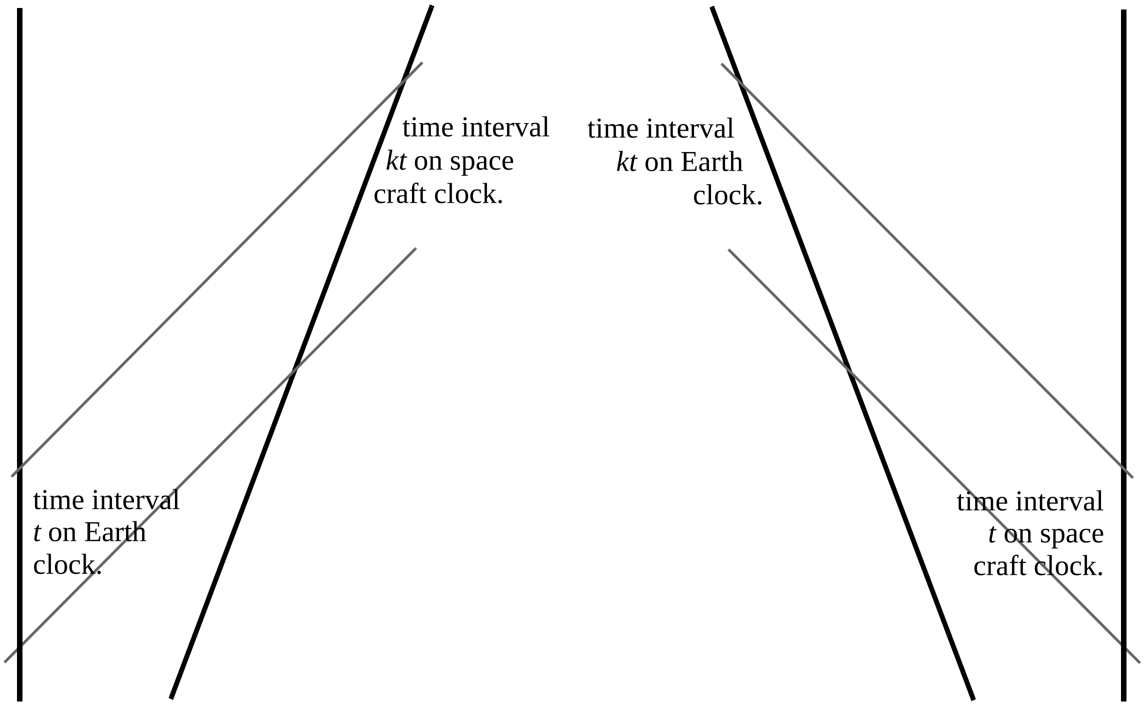


Figure 3: A space craft is uniformly moving in the Earth's reference frame. The space craft and the Earth have identical clocks and communicate with each other by radio or light. The Earth sends the space craft two signals at an interval *t*. The space craft receives them at an interval *kt*. $k \in \mathbb{R}$ is red shift. Although *kt* is not necessarily an integer, its fractional part is less than a chronon, and is lost in measurement.

There is no fundamental difference between the matter in the space craft and the matter in the Earth. The space craft can be regarded as stationary, and the Earth as moving. The principle of homogeneity implies that signals sent by the space craft to the Earth are also subject to red shift. The defining condition for the special theory of relativity is that there is a special class of reference frames such that

Definition: For inertial reference frames redshift is both constant and equal for both observers.

We know from observation, justified by $\mathbb{A}2$, that inertial reference frames exist, at least to the accuracy of measurement, and they will be assumed in this paper. The general theory of relativity places a more general condition on red shift. The implication for quantum space-time will be studied in another paper, currently in draft, in which it is shown that, as a direct result of the discrete nature of particle interactions, the inherent delay in the return of the signal forces the use of non-Lorentzian metric, and results in the force of gravity.

Theorem: Time dilation.

Proof: The space craft and the Earth set both clocks to zero at the moment the space craft passes the Earth. The space craft is moving at speed *v*, so by definition, after time *t* on the Earth clock, the space craft has travelled distance *vt*. For inertial reference frames, if the space craft sends the Earth signals at an interval *t* the Earth receives them at an interval *kt*. Therefore Earth's signal was sent at time *t - vt*, and returned at time *t + vt*. Then by applying the Doppler shift twice, once for the radar sent out and once for it coming back

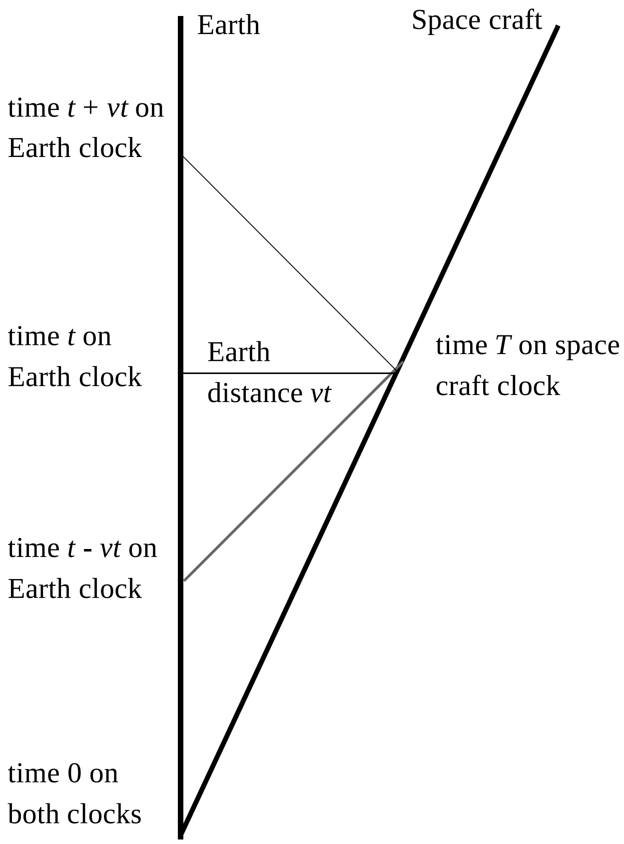
3.4 $$t + vt = k^2(t - vt)$$

According to the Earth the time the signal reaches the space craft is

3.5 $$T = k(t - vt)$$

Eliminating *k* by 3.4 gives the formula for time dilation.

3.6 $$T = t\sqrt{1 - v^2}$$



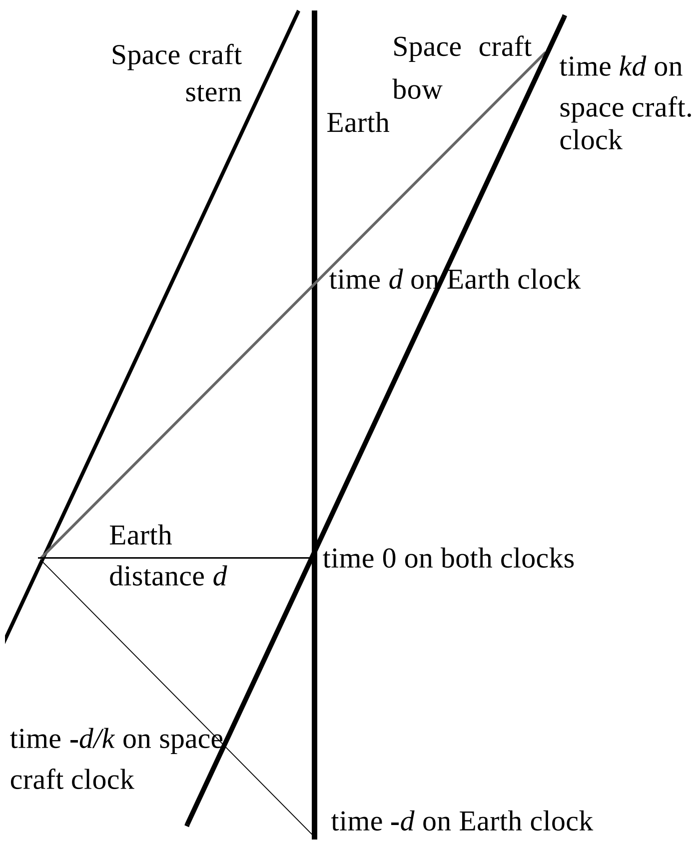
Theorem: Lorentz Contraction.

Proof: The bow and stern of the space craft are shown as parallel lines. The space craft's clock is in the bow. For ease of calculation, both the space craft and the Earth set their clocks to zero when the stern passes the Earth clock. Earth uses radar to measure the distance, *d*, to the bow at time 0. The signal must have been sent at time $-d$, and must return at time *d* on the Earth clock. From the Doppler shift, on the space craft's clock, the signal passes the bow of the space craft at time *-d/k* and comes back to it at time *dk*. So, according to the moving space craft, the length, *D*, of the ship is

3.7 $$D = (dk + d/k)/2$$

Eliminating *k* by 3.4 gives the formula for Lorentz contraction

3.8 $$D = \frac{d}{\sqrt{1 - v^2}}$$



Laws which are the same in all co-ordinate systems are expressed in terms of invariants, mathematical quantities which are the same in all co-ordinate systems. The simplest invariant is an ordinary number or scalar. Another invariant, familiar from classical mechanics, is the vector. Changing the co-ordinate system has no effect on a vector, but it changes the description of a vector in a co-ordinate system.

Definition: A space-time vector is the difference in the co-ordinates of two events. When no ambiguity arises space-time vectors are simply called vectors.

Theorem: The mass shell condition

3.9 $$m^2 = E^2 - \boldsymbol{p}^2$$

Proof: A vector can be represented as a straight line on a space-time diagram, and described by components

3.10 $$r = (E, \boldsymbol{p})$$

For a time-like vector, r, there is a particular reference frame in which it represents a state of rest, namely when it is aligns with the axis representing the clock on which the definition of that reference frame is based. In this reference frame r has co-ordinates

3.11 $$r = (m, 0)$$

An observer moving at velocity $\boldsymbol{v}$ relative to the clock describes r by co-ordinates given by the formulae for time dilation, 3.6 and Fitzgerald contraction, 3.8

3.12 $$r = (E, \boldsymbol{p}) = \left(\frac{m}{\sqrt{1-\boldsymbol{v}^2}}, \frac{m\boldsymbol{v}}{\sqrt{1-\boldsymbol{v}^2}}\right)$$

The mass shell condition, 3.9, follows at once

Definition: If $x = (x_0, x_1, x_2, x_3)$ and $y = (y_0, y_1, y_2, y_3)$ are vectors in space-time then the scalar product is

3.13 $$x \cdot y = -x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3$$

Theorem: The scalar product is invariant under Lorentz transformation

Proof: Straightforward algebra from 3.12

An additional law is needed to extend the definition of space-time co-ordinates to events not directly measured by radar. We express it in a form which encompasses measurements other than time and position.

$\mathbb{L}6$. *When a configuration of matter gives rise to a measurable property, this occurs because the net behaviour of the particles in that configuration generates the properties of the measurement.*

Reason: This is an application of $\mathbb{L}3$. The same interactions between particles take place within the structure of the measuring apparatus as elsewhere, and generate the same types of relationship.

For example the interactions (in this case the exchange of photons) which give rise to geometry in relationships derived from radar, also generate geometry whenever the same interactions takes place. Thus, according to $\mathbb{L}6$, measurement does not create the property of position, but the property of position is the result of an objective configuration of particle interactions, and exists without measurement whenever there is a suitable configuration of particles. Thus $\mathbb{L}6$ extends the properties of the co-ordinate system beyond measurements which are actually carried out, to all situations in which the concept of co-ordinates is meaningful. $\mathbb{L}6$ can also be applied to any situation too complex for analysis, where we understand that the laws found through observation are the consequence of deeper, known laws.

It does not appear that $\mathbb{L}1$ - $\mathbb{L}6$ are sufficient to derive the whole behaviour of matter. In addition we need to know about the actual interactions which take place between particles. It will be seen that the mathematical description of interactions severely constrains their form, but since there is more than one force in nature the interaction is not uniquely determined by $\mathbb{L}1$ - $\mathbb{L}6$. The interaction is also an underlying assumption of the theory, but its description requires a considerable mathematical construction. For completeness in the assumptions, we need a law enabling the justification of a particular choices of interaction

$\mathbb{L}7$. *If it is possible to deduce laws of physics, without theoretical or empirical contradiction, from the mathematical expression of an interaction, then that interaction takes place in nature.*

$\mathbb{L}7$ leaves a certain amount to be desired. Discounting arbitrary variations and complications in mathematics which have no effect on physical law, it may be that the same laws of physics can be deduced from different possible interactions, and that only one of those interactions takes place in nature. If so $\mathbb{L}7$ does not determine which one takes place. This seems extremely unlikely but we have no proof. It may also be that the theory is already so constrained that the only interactions possible on theoretical grounds are the ones which take place in nature; $\mathbb{L}7$ would then not be independent. This also seems unlikely, but other interactions which may be valid in the mathematics may be unsatisfactory on some other grounds.

4 Uncertainty, Probability and Measurement

As geometrical ideas break down, we still have an intuitive idea that the particle has some sort of position, reflected in the fact that if we do carry out a measurement of position we always get a precise result in the form of a number. Many valued logics were introduced in the 1920s, notably by Lukasiewicz, for dealing with the intuitive idea of degrees of certainty, and key ideas in the use and interpretation of many valued logic were described by Max Black, [10]. An excellent and comprehensive survey of many valued logics has been prepared by Rescher [11].

In classical, or crisp, logic, the truth of a proposition is given by the values 0 (definitely false) or 1 (definitely true). Often (but not always) the logical structure is set up in such a way that the truth value of a proposition corresponds to the physical truth of the proposition. In a typical many valued logic a real valued function, $f_P(x)$, is used as a measure of the certainty of the truth of a proposition $P[x]$. For example, fuzzy logic, created by Professor Lofti Zadeh, [12][13], has been used with considerable success in systems science for approximate reasoning based on imprecise information as is typically supplied by natural language, and has developed into a major subject area [14][15]. In fuzzy logic propositions are given real truth values between 0 and 1, and the truth value represents a subjective measure of certainty of the truth of the proposition. Thus the fuzzy representation of the truth of the proposition “Joe Bloggs is a tall man” is a real function of height taking values between 0 and 1, and the truth value of the proposition “Joe Bloggs is 5’11” does not correspond the (unknown) objective truth of the same proposition.

In crisp logic the position of a point can be regarded as a mapping from $\mathbb{N}^3$ to $\{0,1\}$. This mapping confers the truth value 0 or 1 on any statement of the position of the point, so each statement of position is certainly true or certainly false. In many valued logic, other truth values are possible. For example in an infinite valued logic, such as fuzzy logic, the position of a point is a mapping from $\mathbb{R}^3$ to the real interval, [0,1], which expresses the level of certainty of each statement of position. Similarly quantum logic takes into consideration the empirical principle that measurement gives imprecise information as to the nature of objective reality. In quantum logic we use complex truth values. Thus, the quantum position of a particle will be defined (below) as a mapping from $\mathbb{N}^3$ to $\mathbb{C}$ expressing the level of certainty that a measurement of position will produce a given result.

The justification for using complex truth values is that quantum position does not describe objective reality, but is a mathematical device, and like $\sqrt{-1}$, has no direct physical meaning. The quantum position of a particle has a value at each co-ordinate, but uncertainty in position does not imply that the particle is physically spread across co-ordinate space; quantum position is simply the set of truth values describing the level of certainty of finding the particle at each co-ordinate.

When we carry out measurement we set up many repetitions of the system, and record the frequency of each result. Probability is simply a prediction of frequency, so a mathematical model of physics must generate a probability for each possible result. Experiments to determine the behaviour of matter are based on knowledge of the initial state and measurement of the final state. We require laws of physics to predict the change taking place between the first measurement and the second. Although there may be a practical difference between an initial measurement and a final one, both are treated as simply measurements and described formally in the same way.

Although the word measurement is used, it is not taken necessarily to imply measurement *of* anything. Measurement is simply the generation of a value out of the combined interactions of particle and apparatus, and does not necessarily imply that that value exists prior to measurement. When we speak of performing a measurement by the apparatus on the particle we artificially separate the two parts of a physical process. Clearly we cannot carry out measurement of a particle in isolation, or measurement with an apparatus and no particle. A particle (or subsystem) under study and the apparatus used to study it are strictly a single system consisting of many interacting particles.

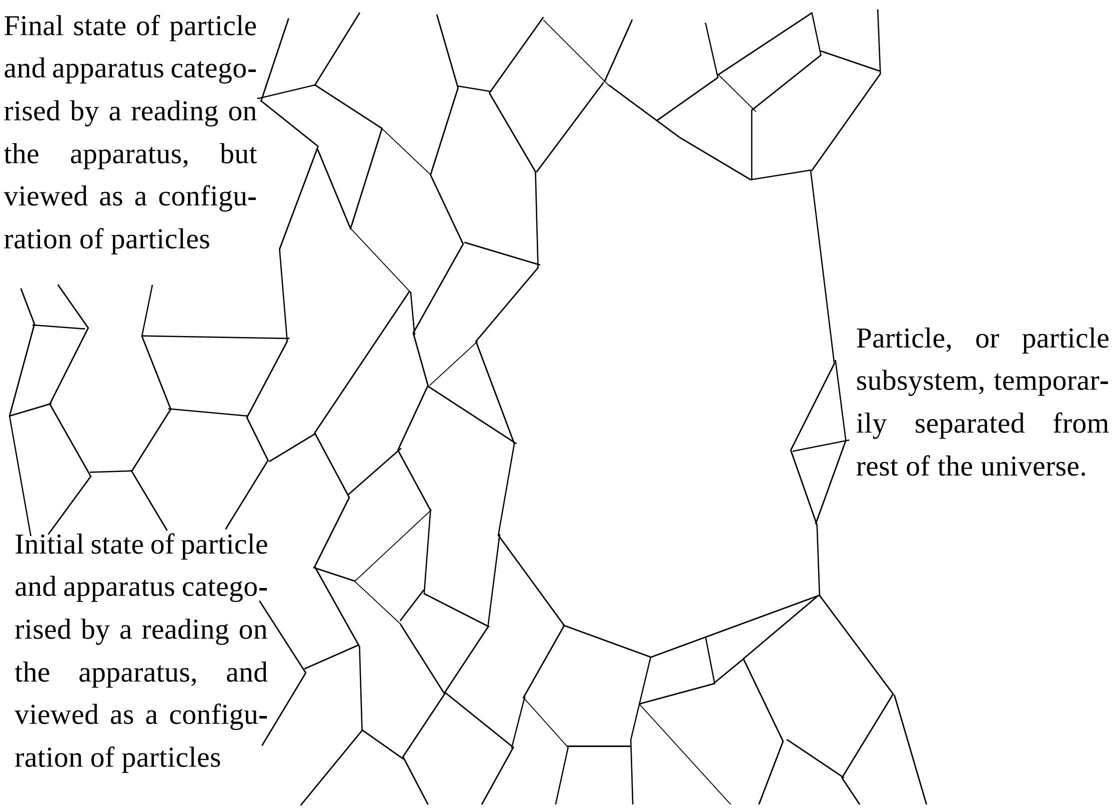


Figure 6. Each line represents the time-line of a particle, and each node represents an interaction. The space between the lines has no meaning, and the only relations permitted between particles are those generated by interactions. As little as possible is assumed about the nature of interactions, except that we assume that they are such that measurements can be carried out. Measurement of a property results in a definite value of that property. This value is used to label the state of particle and apparatus which generated it.

Definition: The ket $| f \rangle$ is a label for a state of particle and apparatus, as categorised by the result, *f*, of measurement. A bra is an alternative representation of a ket.

Kets are labels associated with physical states. This is significant because when we introduce the laws of vector space (i.e. the principle of superposition), we will be speaking of the properties of a labelling system, not of objective properties of matter. But, in keeping with common practice, we loosely refer to kets as states. The laws of physics will express relationships between initial states $| f \rangle$ and final states $\langle g |$, described by placing the bra and the ket together to make a braket $\langle f | g \rangle$.

Definition: The braket is the quantum logical truth value describing the degree of certainty that the state labelled $| g \rangle$ will follow from the initial state $| f \rangle$

According to the rules of quantum logic we have

4.1 $\quad |\langle f | g \rangle| = 1 \quad$ if *f* is certain to follow *g*

4.2 $\quad |\langle f | g \rangle| = 0 \quad$ if *f* cannot result from *g*

A central issue in the application of many valued logic is the determination of a truth function suited to the situation under consideration. We now seek further constraints on the braket.

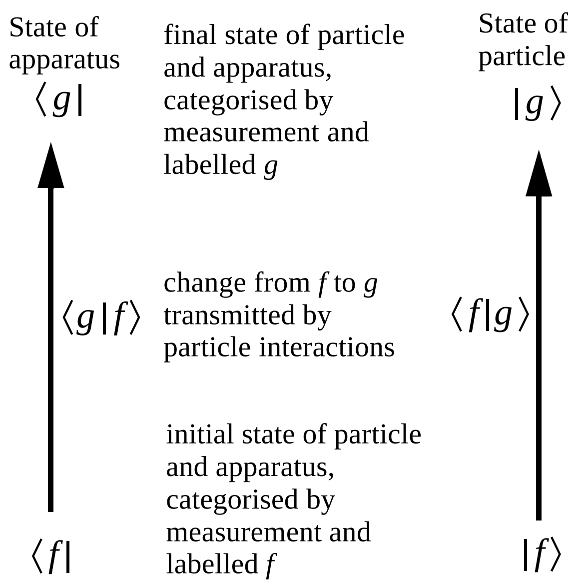


Figure 7. Traditionally in quantum mechanics, kets have been thought of as describing the state of the particle, but what we actually measure is the state of the apparatus. That is to say we read the value of the state from the apparatus, and apply that value to the state of the particle. There is no fundamental difference between the matter in the apparatus and the matter being measured. In spite of the difference in the arrangement of the particles of matter constituting each, they are both labelled by bras and kets. By definition, if the state of the apparatus is categorised by a particular ket, the state of the particle is categorised by the same ket.

The particle alters the state of the apparatus, since we have designed the apparatus to give a reading of the state of the particle. The apparatus also alters the state of the particle, since it is impossible to measure the particle without interacting with it. Whatever the actual configuration of matter, both states are categorised by the same value derived from measurement, ensuring correspondence between state of particle and state of apparatus. Because the labelling of the particle and the labelling of the apparatus are identical, we adopt a definition of a truth value such that the uncertainty in the apparatus is equal to the uncertainty in the position of the particle. We can regard the ket as labelling the state of the measuring apparatus and the bra as labelling the state of the particle, so that, if the apparatus is in the (known) state $| f \rangle$, then the particle is labelled by the bra $\langle f |$ Then, the degree of certainty for a transition of the apparatus to the state $\langle g |$ is $\langle g | f \rangle$, and the corresponding degree of certainty for the particle is $\langle f | g \rangle$. By $\mathbb{L}3$, uncertainty is divided equally between particle and apparatus. So a consistent definition of the braket is constrained to factorise probability

4.3 $\quad$ *Probability*(*g* leads to *f*) $= \langle f | g \rangle \langle g | f \rangle$

4.3 is a defining mathematical relationship, not a physical statement about what actually happens.

Probability is a real valued function so

4.4 $$\langle f | g \rangle = \overline{\langle g | f \rangle}.$$

It is worth remarking that a failing of the Copenhagen interpretation is that it describes the particle with an uncertainty relation and the apparatus as certain. Here uncertainty is divided equally between particle and apparatus, but uncertainty in the apparatus is less in relation to its size.

5 Ket Space

Assume only measurement of position of a single elementary particle of a given type. Correspondence between kets and other types of measurement will be constructed, as will the remaining laws of quantum mechanics. In chronons, the result of a measurement of time and position is a point in $\mathbb{N}^3$. In practice there is also a bound on the magnitude of the result, so we may take the results of measurement of position to be in a finite region $\mathrm{N} \subset \mathbb{N}^3$ N is not a bound on the size of the universe and merely has to be large enough to be able to say with certainty that N contains any particle under study, i.e. the quantum position function of the particle vanishes outside of N. Without loss of generality define

Definition: The coordinate system is $\mathrm{N} = (\nu, \nu) \otimes (\nu, \nu) \otimes (\nu, \nu) \subset \mathbb{N}^3$ for some $\nu \in \mathbb{N}$.

Definition: for any point $\boldsymbol{x} \in \mathrm{N}$, $|\boldsymbol{x}\rangle$ is the ket corresponding to a measurement of position with result $\boldsymbol{x}$. $|\boldsymbol{x}\rangle$ is called a position ket.

Definition: Let $\mathbb{H}_0$ be the set of kets resulting from a measurement of position of an elementary particle in N.

$\mathbb{H}_0$ contains kets for all physically realised measurements of position, but also kets for measurements of position which may be made in principle, and it also contains kets which may not be realised either in principle or in practice.

Definition: Construct a vector space, $\mathbb{H}$, over $\mathbb{C}$, with basis $\mathbb{H}_0$

Remark: This is trivial because $\mathbb{H}_0$ is finite. $\mathbb{H}$ has dimension $(2\nu\text{-}1)^3$.

Vector space introduces intuitive logical operations between uncertain propositions. Addition corresponds to logical OR, and multiplication by a scalar gives an intuitive idea of weighting due to the level of certainty in each option given to logical OR. This is justified because kets are labels for possible states of matter, not descriptions of reality. Thus the principle of superposition is a definitional truism, not a physical assumption. Vector space extends the labelling system from $\mathbb{H}_0$ to $\mathbb{H}$. Because multiplication by a scalar only has meaning as a weighting between alternatives, $\forall |f\rangle \in \mathbb{H}$, $\forall \lambda \in \mathbb{C}$, such that$\lambda \neq 0$ $\lambda|f\rangle$ is a label for the same physical state as $|f\rangle$. We can therefore renormalise kets as we choose, without affecting their use as labels for states.

In a measurement of position a particle can be found anywhere, but it is only found in one place at a time. The braket, or quantum position, which describes this is a Kronecker delta, renormalised to

5.1 $$\forall \boldsymbol{x}, \boldsymbol{y} \in \mathrm{N}, \langle \boldsymbol{x} | \boldsymbol{y} \rangle = \chi^3 \delta_{\boldsymbol{xy}}$$

Definition: With this normalisation, the quantum position of a particle in the state $|f\rangle \in \mathbb{H}$ is the function $\mathrm{N} \to \mathbb{C}$ defined by

5.2 $$\forall \boldsymbol{x} \in \mathrm{N}, \boldsymbol{x} \to \langle \boldsymbol{x} | f \rangle$$

It will found that quantum position at any time can be identified with the restriction of the wave function to $\mathbb{N}^3$. We will use the term quantum position, to emphasise its meaning in quantum logic, because it is discrete, and because a wave equation is not assumed.

From the property that any vector can be expanded in terms of a basis we have

$$\forall |f\rangle \in \mathbb{H}, \exists f : \mathrm{N} \to \mathbb{C} \text{ such that } |f\rangle = \sum_{\boldsymbol{x} \in \mathrm{N}} \frac{1}{\chi^3} f(\boldsymbol{x}) |\boldsymbol{x}\rangle$$

By applying $\langle x|$ to both sides and using 5.1 we have $f(\boldsymbol{x}) = \langle \boldsymbol{x}|f\rangle$, so

5.3 $$\forall |f\rangle \in \mathbb{H}, |f\rangle = \sum_{\boldsymbol{x} \in \mathrm{N}} \frac{1}{\chi^3} |\boldsymbol{x}\rangle\langle \boldsymbol{x}|f\rangle$$

5.3 is true for all $|f\rangle$, and hence we can define an operator expression known as the resolution of unity

5.4 $$\sum_{\boldsymbol{x} \in \mathrm{N}} \frac{1}{\chi^3} |\boldsymbol{x}\rangle\langle \boldsymbol{x}| = 1$$

So the braket is given by

5.5 $$\langle g|f\rangle = \sum_{\boldsymbol{x} \in \mathrm{N}} \frac{1}{\chi^3} \langle g|\boldsymbol{x}\rangle\langle \boldsymbol{x}|f\rangle$$

which is the hermitian form known as the scalar product. There is a homomorphic correspondence between $\mathbb{H}$ and the space of complex functions on N given by the correspondence between a ket and its quantum position function. Quantum position can also be regarded as the set of components of a vector in a particular basis.

6 Quantum Paradox

If ket space is understood as a labelling system constructed by an individual observer from the available information, the familiar paradoxes of quantum mechanics do not arise. No physical process is described by the collapse of the wave function, but the measurement provides the observer with additional information, and enables him to recategorise the state. Schrödinger's cat [16] is labelled as a quantum mixture of live and dead states, because there is insufficient information to say which, but there is no implication that the ontological cat is other than strictly alive or strictly dead. Similarly, Wigner and his friend [17] have different information, and construct different labels for the same objective situation.

The EPR paradox [18] does not indicate a faster than light physical process, or even a physical entanglement of spacially separated particles, merely an entanglement of labels such that knowledge of the state of one particle allows us to relabel the state of the other. This is of no benefit to the observer of the other particle, and does not allow him to relabel the state until such time as we can inform him of our result, at a speed less than that of light.

In the classical Young's slits experiment where a particle passes through a pair of slits and hits a screen, the question of which slit the particle passes through can be understood as being incorrectly phrased. If geometrical relationships actually arise from interactions, then the absence of interaction implies the absence of geometrical relationships. It is in the essence of the experiment that the particle does not interact with other matter between the instant of its emission from the source and the instant when it hits the screen. It follows that the particle does not have a clear geometrical relationship with the slits, so that the question "Which slit does it pass through?" does not make sense.

The violation of the Bell inequalities [19] as shown by the Aspect experiments [20] is more subtle, since apparently the manner in which one observer carries out his measurement does physically affect the result of the other measurement. According to Bell, the proof of Bell's inequality depends on a) realism b) determinism and c) locality. Realism is $\mathbb{A}1$, and we are not prepared to discard it. Although a locality condition, 18.19, will be established below, and we establish the point-like nature of particles in section 19, this is not locality in the sense intended by Bell. The absence of interaction between the creation of a pair of particles and their detection implies an absence of geometrical relationships so we cannot actually say that they are geometrically separated until they are detected.

In spite of the apparent conflict between determinism and free will, physicists are sometimes thought to be reluctant to drop determinism. But there is no à prioré reason to believe in causality from past to future, and although time symmetry is broken by the statistical law of entropy, at a fundamental level the laws of physics are time (or rather PCT) symmetric. So we do not exclude the possibility that the detection of the polarised of one particle may alter the polarisation of both particles at their creation.

7 Momentum Space

Definition: Momentum space is $\mathrm{M} = (-\pi, \pi] \otimes (-\pi, \pi] \otimes (-\pi, \pi]$; the elements of momentum space are called momenta.

Definition: For each value of momentum $\boldsymbol{p} \in \mathrm{M}$, define a ket $|\boldsymbol{p}\rangle$, known as a plane wave state, by the quantum position

7.1 $$\langle \boldsymbol{x} | \boldsymbol{p} \rangle = \left(\frac{\chi}{2\pi}\right)^{\frac{3}{2}} e^{-i\boldsymbol{x} \cdot \boldsymbol{p}}$$

It will be shown later that $\boldsymbol{p}$ gives rise to classical momentum. The expansion of $|\boldsymbol{p}\rangle$ in the basis $\mathbb{H}_0$ is calculated by using the resolution of unity, 5.4

7.2 $$|\boldsymbol{p}\rangle = \sum_{\boldsymbol{x} \in \mathrm{N}} \frac{1}{\chi^3} |\boldsymbol{x}\rangle\langle \boldsymbol{x} | \boldsymbol{p}\rangle = \left(\frac{\chi}{2\pi}\right)^{\frac{3}{2}} \sum_{\boldsymbol{x} \in \mathrm{N}} \frac{1}{\chi^3} e^{-i\boldsymbol{x} \cdot \boldsymbol{p}} |\boldsymbol{x}\rangle$$

Definition: For each ket $|f\rangle$ define the Fourier transform, $F : \mathrm{M} \to \mathbb{C}$, also called the momentum space function

7.3 $$F(\boldsymbol{p}) = \langle \boldsymbol{p} | f \rangle$$

Then, by 5.4, F can be expanded as a trigonometric polynomial

$$F(\boldsymbol{p}) = \sum_{\boldsymbol{x} \in \mathrm{N}} \frac{1}{\chi^3} \langle \boldsymbol{p} | \boldsymbol{x} \rangle \langle \boldsymbol{x} | f \rangle$$

7.4 $$= \left(\frac{\chi}{2\pi}\right)^{\frac{3}{2}} \sum_{\boldsymbol{x} \in \mathrm{N}} \frac{1}{\chi^3} \langle \boldsymbol{x} | f \rangle e^{i\boldsymbol{x} \cdot \boldsymbol{p}} \qquad \text{by 7.1 and 7.3}$$

Lemma:

7.5 $$\forall x, y \in \mathbb{N}, \quad \int_{-\pi}^{\pi} dp e^{-i(x-y) \cdot p} = \begin{cases} 2\pi & \text{if } y = x \\ 0 & \text{otherwise} \end{cases}$$

Proof: Straightforward trigonometry.

Clearly the cardinality of the plane wave states is greater than the cardinality of $\mathbb{H}_0$, so plane waves are not a basis. But quantum position can be found in terms of plane waves from the Fourier coefficient

$$\left(\frac{\chi}{2\pi}\right)^{\frac{3}{2}} \int_{\mathrm{M}} d^3\boldsymbol{p} F(\boldsymbol{p}) e^{-i\boldsymbol{x} \cdot \boldsymbol{p}} = \frac{\chi^3}{8\pi^3} \int_{\mathrm{M}} d^3\boldsymbol{p} \sum_{\boldsymbol{y} \in \mathrm{N}} \frac{1}{\chi^3} \langle \boldsymbol{x} | f \rangle e^{i\boldsymbol{y} \cdot \boldsymbol{p}} e^{-i\boldsymbol{x} \cdot \boldsymbol{p}} \qquad \text{by 7.4 and 7.1}$$

7.6 $$= \langle \boldsymbol{x} | f \rangle \qquad \text{by 7.5}$$

Rewriting 7.6 in the notations of 7.3 and 7.1

7.7 $$\langle \boldsymbol{x} | f \rangle = \int_{\mathrm{M}} d^3\boldsymbol{p} \langle \boldsymbol{x} | \boldsymbol{p} \rangle \langle \boldsymbol{p} | f \rangle$$

Then $\forall | f \rangle, | g \rangle \in \mathbb{H}$

$$\langle g | f \rangle = \sum_{\boldsymbol{x} \in \mathbb{N}} \frac{1}{\chi^3} \langle g | \boldsymbol{x} \rangle \langle \boldsymbol{x} | f \rangle \qquad \text{by 5.4}$$

$$= \int_{\mathrm{M}} d^3\boldsymbol{p} \sum_{\boldsymbol{x} \in \mathbb{N}} \frac{1}{\chi^3} \langle g | \boldsymbol{x} \rangle \langle \boldsymbol{x} | \boldsymbol{p} \rangle \langle \boldsymbol{p} | f \rangle \qquad \text{by 7.7}$$

7.8 $$= \int_{\mathrm{M}} d^3\boldsymbol{p} \langle g | \boldsymbol{p} \rangle \langle \boldsymbol{p} | f \rangle \qquad \text{by 5.4}$$

7.8 is true for all $| f \rangle$ and $| g \rangle$, and hence we can define a second operator expression known as the resolution of unity

7.9 $$\int_{\mathrm{M}} d^3\boldsymbol{p} | \boldsymbol{p} \rangle \langle \boldsymbol{p} | = 1$$

It follows immediately that

7.10 $$\langle \boldsymbol{q} | f \rangle = \int_{\mathrm{M}} d^3\boldsymbol{p} \langle \boldsymbol{q} | \boldsymbol{p} \rangle \langle \boldsymbol{p} | f \rangle$$

So $\langle \boldsymbol{p} | \boldsymbol{q} \rangle$ has the effect of a Dirac delta function on the test space of momentum space functions.

Definition: The delta function is

7.11 $$\delta : \mathrm{M} \to \mathbb{C} \qquad \delta(\boldsymbol{p} - \boldsymbol{q}) = \langle \boldsymbol{q} | \boldsymbol{p} \rangle$$

Explicitly, calculating $\langle \boldsymbol{p} | \boldsymbol{q} \rangle$ directly from 7.2

7.12 $$\delta(\boldsymbol{p} - \boldsymbol{q}) = \frac{1}{8\pi^3} \sum_{\boldsymbol{x} \in \mathrm{N}} e^{-i\boldsymbol{x} \cdot (\boldsymbol{p} - \boldsymbol{q})}$$

The lack of symmetry between momentum space and co-ordinate space reflects the idea that position is closely associated with the fundamental nature of matter, whereas momentum is a mathematical construction. The dependency of momentum space functions on N is irrelevant since it only effects kets with quantum positions which exhibit a sharp cutoff at the boundary of N. These are not considered here, and it is always possible to exclude them by increasing the value of ν. N is bounded so it is not possible to define unlimited space translation, but N is large enough to contain any particles under study, and can be taken larger without loss of generality. Under a space translation, $\boldsymbol{z}$, of the co-ordinate system such that the particles under consideration certainly remain in N, 7.6 becomes

7.13 $$\forall \boldsymbol{x} \in \mathrm{N} \qquad \langle \boldsymbol{x} - \boldsymbol{z} | f \rangle = \begin{cases} \dfrac{\chi^3}{8\pi^3} \displaystyle\int_{\mathrm{M}} d^3\boldsymbol{p} F(\boldsymbol{p}) e^{i\boldsymbol{z} \cdot \boldsymbol{p}} e^{-i\boldsymbol{x} \cdot \boldsymbol{p}} & \text{if} \quad \boldsymbol{x} - \boldsymbol{z} \in \mathrm{N} \\ 0 & \text{otherwise} \end{cases}$$

By 7.8, multiplication of the momentum space functions by $e^{i\boldsymbol{p} \cdot \boldsymbol{z}}$ is a homomorphic correspondence, and by 7.13 it is equivalent to space translation, $\boldsymbol{z}$, of the co-ordinate system in the subspace of kets for states of particles which are certainly in N both before and after the translation.

8 Multiparticle States

Definition: The vector space, $\mathbb{H}^n$, of kets for labelling multiparticle states of particles of the same type is defined by

i. $\mathbb{H}^0 = \{\lambda|\rangle : \lambda \in \mathbb{C}\}$ i.e. the space containing only the empty ket, a label for a state of no particles, (the vacuum state). It is trivial that $\mathbb{H}^0$ is a one dimensional vector space isomorphic to $\mathbb{C}$, so we can identify $\mathbb{H}^0 = \mathbb{C}$. The empty ket is normalised by 4.1 to

8.1 $$\langle | \rangle = 1$$

ii $\mathbb{H}^1 = \mathbb{H}^0 \oplus \mathbb{H}$ Clearly a one particle state cannot be a no particle state, so by the definition of the braket as a measure of uncertainty

8.2 $$\forall\, |f\rangle \in \mathbb{H} \quad \langle | f \rangle = 0$$

iii. For $n \in \mathbb{N}, n > 0$ $\mathbb{H}^n = \bigotimes_n \mathbb{H}^1$ (the external direct product).

Thus the elements of $\mathbb{H}^n$ are ordered *n*-tuples such that addition is given by

8.3 $$(|f_1\rangle, \dots, |f_n\rangle) + (|g_1\rangle, \dots, |g_n\rangle) = (|f_1\rangle + |g_1\rangle, \dots, |f_n\rangle + |g_n\rangle)$$

and multiplication by a scalar $\lambda \in \mathbb{C}$ is given by

8.4 $$\lambda(|f_1\rangle, \dots, |f_n\rangle) = (\lambda|f_1\rangle, \dots, \lambda|f_n\rangle)$$

For the states $|f\rangle = (|f_1\rangle, \dots, |f_n\rangle)$ and $|g\rangle = (|g_1\rangle, \dots, |g_n\rangle)$ the braket is given by

8.5 $$\langle |f_1\rangle, \dots, |f_n\rangle \,|\, |g_1\rangle, \dots, |g_n\rangle \rangle = \prod_{i=1}^{n} \langle f_i | g_i \rangle$$

which is required by 4.3 for the interpretation that each of the particles is independent. Hence, by 5.1, $\forall \boldsymbol{x}^i \in \mathrm{N}$, $i = 1, \dots, n$ the basis $(|\boldsymbol{x}^1\rangle, \dots, |\boldsymbol{x}^n\rangle)$ is normalised such that

8.6 $$\langle |\boldsymbol{y}^1\rangle, \dots, |\boldsymbol{y}^n\rangle \| \boldsymbol{x}^1\rangle, \dots, |\boldsymbol{x}^n\rangle\rangle = \chi^{3n} \prod_{i=1}^{n} \delta_{\boldsymbol{y}^i \boldsymbol{x}^i}$$

Definition: Let $\mathbb{H}_0^n = \bigotimes_n \mathbb{H}_0 \cup \mathbb{H}^0$. Clearly $\mathbb{H}_0^n$ is a basis of $\mathbb{H}^n$.

Definition: The space of all particles of the same type is $\mathbb{H}^N$ where $N \in \mathbb{N}$ is larger than the number of particles in the universe.

The statement that we can take a value of *N* greater than any given value is the definition of an infinite sequence, so, in effect the space of all particles of the same type is $\mathbb{H}^\infty$

Corollary: $\forall i, n \in \mathbb{N}$, such that $0 < i < n$, $\mathbb{H}^i \subset \mathbb{H}^n$ is an isomorphic embedding under the mapping

8.7 $$\mathbb{H}^i \to \mathbb{H}^n : \quad (|f_1\rangle, \dots, |f_i\rangle) \to (|f_1\rangle, \dots, |f_i\rangle, |\rangle, \dots, |\rangle)$$

Definition: The space of all particles is $\mathscr{H} = \bigotimes_\gamma \mathbb{H}_\gamma^\infty$ where γ runs over every type of particle.

Corollary: It follows immediately that

8.8 $$\forall n \in \mathbb{N}, \quad \mathbb{C} = \mathbb{H}^0 \subset \mathbb{H}^n \subset \mathscr{H}$$

Until the treatment of interactions, we ignore states of different types of particle, for which $\langle f | g \rangle = 0$.

9 Creation Operators

The creation of a particle in an interaction is described by the action of a creation operator. Creation operators incorporate the idea that particles of the same type are identical, so that when a particle is created it is impossible to distinguish it from any existing particle of the same type. They are defined by their action on the basis of $\mathbb{H}^1$. The definition removes arbitrary phase and normalises the two particle state to coincide with 8.6.

Definition: $\forall |\boldsymbol{x}\rangle \in \mathbb{H}_0^1$ the creation operator $|\boldsymbol{x}\rangle$ is defined by $\forall |\boldsymbol{y}\rangle \in \mathbb{H}_0^1$

$$|\boldsymbol{x}\rangle : |\boldsymbol{y}\rangle \to |\boldsymbol{x}\rangle|\boldsymbol{y}\rangle = |\boldsymbol{x};\boldsymbol{y}\rangle$$

9.1 $$= \tfrac{1}{\sqrt{2}}[(|\boldsymbol{x}\rangle, |\boldsymbol{y}\rangle) + \kappa(|\boldsymbol{y}\rangle, |\boldsymbol{x}\rangle)]$$

where $\kappa \in \mathbb{C}$ is to be determined.

Definition: The bra corresponding to $|\boldsymbol{x};\boldsymbol{y}\rangle \in \mathbb{H}^2$ is designated by $\langle \boldsymbol{x};\boldsymbol{y}| \in \mathbb{H}^2$

Now, by 8.5 and 9.1 $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathrm{N}$

$$\langle \boldsymbol{x};\boldsymbol{y}|\boldsymbol{x};\boldsymbol{y}\rangle = \tfrac{1}{2}[\langle \boldsymbol{x}|\boldsymbol{x}\rangle\langle \boldsymbol{y}|\boldsymbol{y}\rangle + \kappa^2\langle \boldsymbol{x}|\boldsymbol{x}\rangle\langle \boldsymbol{y}|\boldsymbol{y}\rangle + 2\kappa\langle \boldsymbol{x}|\boldsymbol{y}\rangle\langle \boldsymbol{y}|\boldsymbol{x}\rangle]$$

9.2 $$= \tfrac{1}{2}[(1+\kappa^2)\chi^6 + 2\kappa\delta_{\boldsymbol{xy}}^2]$$

The order in which particles are created can make no difference to the state, so

9.3 $$\exists\, \lambda \in \mathbb{C} \text{ such that } |\boldsymbol{x};\boldsymbol{y}\rangle = \lambda|\boldsymbol{y};\boldsymbol{x}\rangle$$

Thus, by direct application of 8.5 and 9.1

$$\langle \boldsymbol{x};\boldsymbol{y}|\boldsymbol{x};\boldsymbol{y}\rangle = \lambda\langle \boldsymbol{x};\boldsymbol{y}|\boldsymbol{y};\boldsymbol{x}\rangle$$

$$= \tfrac{1}{2}\lambda[\kappa\langle \boldsymbol{x}|\boldsymbol{x}\rangle\langle \boldsymbol{y}|\boldsymbol{y}\rangle + \kappa\langle \boldsymbol{x}|\boldsymbol{x}\rangle\langle \boldsymbol{y}|\boldsymbol{y}\rangle + (1+\kappa^2)\langle \boldsymbol{x}|\boldsymbol{y}\rangle\langle \boldsymbol{y}|\boldsymbol{x}\rangle]$$

9.4 $$= \tfrac{1}{2}\lambda[2\kappa\chi^6 + (1+\kappa^2)\delta_{\boldsymbol{xy}}^2]$$

Comparison of 9.2 with 9.4 gives

9.5 $$1+\kappa^2 = 2\lambda\kappa \qquad \text{and} \qquad \lambda(1+\kappa^2) = 2\kappa$$

Hence $\lambda^2 = 1$. $\lambda = \pm 1$. Substituting into 9.5;

if $\lambda = -1$, then $1+\kappa^2 = -2\kappa$ so $\kappa = -1$;

if $\lambda = 1$, then $1+\kappa^2 = 2\kappa$ so $\kappa = 1$

Definition: Bosons are particles for which $\kappa = 1$, so that $\forall |\boldsymbol{x}\rangle \in \mathbb{H}_0$ the creation operators $|\boldsymbol{x}\rangle$ obey

9.6 $$\forall\, \boldsymbol{y} \in \mathrm{N} \qquad |\boldsymbol{x};\boldsymbol{y}\rangle = \tfrac{1}{\sqrt{2}}[(|\boldsymbol{x}\rangle, |\boldsymbol{y}\rangle) + (|\boldsymbol{y}\rangle, |\boldsymbol{x}\rangle)] = |\boldsymbol{y};\boldsymbol{x}\rangle$$

Definition: Fermions are particles for which $\kappa = -1$, so that $\forall |\boldsymbol{x}\rangle \in \mathbb{H}_0$ the creation operators obey

9.7 $$\forall\, \boldsymbol{y} \in \mathrm{N} \qquad |\boldsymbol{x};\boldsymbol{y}\rangle = \tfrac{1}{\sqrt{2}}[(|\boldsymbol{x}\rangle, |\boldsymbol{y}\rangle) - (|\boldsymbol{y}\rangle, |\boldsymbol{x}\rangle)] = -|\boldsymbol{y};\boldsymbol{x}\rangle$$

The use of the ket notation for creation operators is justified by the homomorphism defined by

9.8 $$|\boldsymbol{x}\rangle|\,\rangle = \tfrac{1}{\sqrt{2}}[(|\boldsymbol{x}\rangle, |\,\rangle) + \kappa(|\,\rangle, |\boldsymbol{x}\rangle)]$$

It is straightforward to check that this is a homomorphism with the scalar product defined by 8.5. In general the creation operator is defined by linearity

9.9 $$\forall\, |f\rangle \in \mathbb{H} \qquad |f\rangle : \mathbb{H}^1 \to \mathbb{H}^2,\ |f\rangle = \sum_{\boldsymbol{x} \in \mathrm{N}} \langle \boldsymbol{x}|f\rangle|\boldsymbol{x}\rangle$$

It follows immediately that $\forall |f\rangle, |g\rangle \in \mathbb{H}$

$$|f;g\rangle = |f\rangle|g\rangle$$

$$= \sum_{x \in \mathrm{N}} \langle \boldsymbol{x}|f\rangle|\boldsymbol{x}\rangle \sum_{y \in \mathrm{N}} \langle \boldsymbol{y}|g\rangle|\boldsymbol{y}\rangle$$

9.10 $$= \sum_{x, y \in \mathrm{N}} \langle \boldsymbol{x}|f\rangle\langle \boldsymbol{y}|g\rangle|\boldsymbol{x};\boldsymbol{y}\rangle$$

Using 9.10 gives

9.11 $\forall$ Boson $|f\rangle, |g\rangle \in \mathbb{H}^1$ $\qquad |f;g\rangle = |g;f\rangle$

and

9.12 $\forall$ Fermion $|f\rangle, |g\rangle \in \mathbb{H}^1$ $\qquad |f;g\rangle = -|g;f\rangle$

Theorem: The Pauli exclusion principle holds for fermions.
Proof: From 9.12, $\forall$ Fermion $|f\rangle \in \mathbb{H}^1$ $\qquad |f;f\rangle = -|f;f\rangle$. Hence

9.13 $\forall$ Fermion $|f\rangle \in \mathbb{H}^1$ $\qquad |f;f\rangle = 0$

i.e. no two fermions may be in the same state.

The definition of the creation operator extends to $|\boldsymbol{x}\rangle : \mathbb{H}^n \to \mathbb{H}^{n+1}$ by requiring that its action on each particle of an *n* particle state is identical, and that it reduces to 9.1 in the restriction of $\mathbb{H}^n$ to the space of the *i*th particle. Thus $\forall\, \boldsymbol{x}, \boldsymbol{y}^i \in \mathrm{N}, i = 1, \ldots, n$

9.14 $$|\boldsymbol{x}\rangle : (|\boldsymbol{y}^1\rangle, \ldots, |\boldsymbol{y}^n\rangle) \to \frac{1}{\sqrt{n+1}}\Bigg((|\boldsymbol{x}\rangle, |\boldsymbol{y}^1\rangle, \ldots, |\boldsymbol{y}^n\rangle) + \kappa\sum_{i=1}^{n}(|\boldsymbol{y}^i\rangle, |\boldsymbol{y}^1\rangle, \ldots, |\boldsymbol{x}\rangle, \ldots, |\boldsymbol{y}^n\rangle)\Bigg)$$

where $\kappa = 1$ for bosons and $\kappa = -1$ for fermions, and $|\boldsymbol{x}\rangle$ appears in the *i*+1th position in the *i*th term of the sum. The normalisation is determined from 8.6 by observing that when all $\boldsymbol{x}, \boldsymbol{y}^i$ are distinct, the right hand side is the sum of *n*+1 orthogonal vectors, normalised to $\chi^{3(n+1)}$. 9.14 holds $\forall\, |f\rangle \in \mathbb{H}$ and $\forall\, |g\rangle \in \mathbb{H}^n$ by linearity.

Definition: The space of physically realisable states is the subspace $\mathscr{F} \subset \mathscr{H}$ which is generated from $\mathbb{H}^0 = \{|\rangle\}$ by the action of creation operators[1].

Definition: Notation for the elements of $\mathscr{F}$ is defined inductively.

9.15 $\forall\, |g\rangle \in \mathbb{H}^1, \forall\, |f\rangle \in \mathbb{H}^n \cap \mathscr{F}$ $\qquad |g;f\rangle = |g\rangle|f\rangle \in \mathbb{H}^{n+1} \cap \mathscr{F}$

Corollary: $|g;f\rangle$ is identified with the creation operator $\mathscr{H} \to \mathscr{H}$ given by $|g;f\rangle = |g\rangle|f\rangle$

Definition: The bra corresponding to $|g;f\rangle \in \mathbb{H}^{n+1}$ is $\langle g;f|$.

Theorem: $\forall |\boldsymbol{x}^i\rangle \in \mathbb{H}^1_0, i = 1, \ldots, n$

9.16 $$|\boldsymbol{x}^1;\boldsymbol{x}^2;\ldots;\boldsymbol{x}^n\rangle = \frac{1}{\sqrt{n!}}\sum_{\pi}\varepsilon(\pi)(|\boldsymbol{x}^{\pi(1)}\rangle, \ldots, |\boldsymbol{x}^{\pi(n)}\rangle)$$

where the sum runs over all permutations π of (1,2,...,*n*), and $\varepsilon(\pi)$ is the sign of π for fermions and $\varepsilon(\pi)=1$ for bosons.

1. An interesting theory of strong interactions and quark confinement can be constructed with the assumption that creation operators for quarks appear in electroweak interactions only in triplets and in quark antiquark pairs, thus escaping the Pauli exclusion principle for individual quarks. This is the subject of a paper currently in draft.

Proof: By induction, 9.16 holds for $n = 2$, by 9.6 and 9.7. Now suppose that 9.16 holds $\forall n < m \in \mathbb{N}$, then, from definition 9.15

$$|\boldsymbol{x}^1;\boldsymbol{x}^2;\ldots;\boldsymbol{x}^m\rangle = |\boldsymbol{x}^1\rangle|\boldsymbol{x}^2;\ldots;\boldsymbol{x}^m\rangle$$
$$= \frac{1}{\sqrt{(m-1)!}}\sum_{\pi}\varepsilon(\pi)|\boldsymbol{x}^1\rangle(|\boldsymbol{x}^{\pi(2)}\rangle, \ldots, |\boldsymbol{x}^{\pi(m)}\rangle)$$

by the inductive hypothesis. 9.16 follows from application of 9.14.

Corollary: $\forall\ |g\rangle, |f\rangle \in \mathbb{H}_0^1$ the creation operators obey the (anti)commutation relations

9.17 $$[|g\rangle, |f\rangle]_{\pm} = 0$$

where for fermions

9.18 $$[x, y]_{+} = \{x, y\} = xy + yx$$

and for bosons

9.19 $$[x, y]_{-} = [x, y] = xy - yx$$

Proof: By definition 9.15, $\forall \boldsymbol{x}^i \in$ N, $i = 1, \ldots, n$, $\forall |\boldsymbol{x}\rangle, |\boldsymbol{y}\rangle \in \mathbb{H}_0^1$

$$|\boldsymbol{x}\rangle, |\boldsymbol{y}\rangle|\boldsymbol{x}^1;\boldsymbol{x}^2;\ldots;\boldsymbol{x}^n\rangle = |\boldsymbol{x};\boldsymbol{y};\boldsymbol{x}^1;\boldsymbol{x}^2;\ldots;\boldsymbol{x}^n\rangle$$
$$= \kappa|\boldsymbol{y};\boldsymbol{x};\boldsymbol{x}^1;\boldsymbol{x}^2;\ldots;\boldsymbol{x}^n\rangle \qquad \text{by 9.16}$$
$$= \kappa|\boldsymbol{x}\rangle, |\boldsymbol{y}\rangle|\boldsymbol{x}^1;\boldsymbol{x}^2;\ldots;\boldsymbol{x}^n\rangle$$

But by definition the kets $|\boldsymbol{x}^1;\boldsymbol{x}^2;\ldots;\boldsymbol{x}^n\rangle$ span $\mathscr{F}$. So by linearity

9.20 $$[|\boldsymbol{x}\rangle, |\boldsymbol{y}\rangle]_{\pm} = 0$$

9.17 follows from 9.10.

Corollary: $\forall\ |f\rangle \in \mathbb{H}$ the creation operator $|f\rangle$ (anti)commutes with the creation operator $|\rangle$ for the vacuum state

9.21 $$[|\rangle, |f\rangle]_{\pm} = 0$$

Theorem: $\forall |\boldsymbol{x}^i\rangle, |\boldsymbol{y}^i\rangle \in \mathbb{H}_0^1, i = 1, \ldots, n$

9.22 $$\langle\boldsymbol{y}^1;\ldots;\boldsymbol{y}^n|\boldsymbol{x}^1;\ldots;\boldsymbol{x}^n\rangle = \sum_{\pi}\varepsilon(\pi)\prod_{i=1}^{n}\langle\boldsymbol{y}^i|\boldsymbol{x}^{\pi(i)}\rangle$$

Proof: By 9.16 and 8.5

$$\langle\boldsymbol{y}^1;\ldots;\boldsymbol{y}^n|\boldsymbol{x}^1;\ldots;\boldsymbol{x}^n\rangle = \frac{1}{n!}\sum_{\pi'}\varepsilon(\pi')\sum_{\pi''}\varepsilon(\pi'')\prod_{i=1}^{n}\langle\boldsymbol{y}^{\pi'(i)}|\boldsymbol{x}^{\pi''(i)}\rangle$$
$$= \frac{1}{n!}\sum_{\pi'}\varepsilon(\pi')\sum_{\pi\pi'}\varepsilon(\pi\pi')\prod_{i=1}^{n}\langle\boldsymbol{y}^{\pi'(i)}|\boldsymbol{x}^{\pi\pi'(i)}\rangle$$

where we observe that $\forall$ permutations π'', π', $\exists$ a permutation π such that $\pi'' = \pi\pi'$. 9.22 follows since the sum over π' contains $n!$ terms which are identical up to the ordering of the factors in the product.

Corollary: $\forall\ |g_i\rangle, |f_j\rangle \in \mathbb{H}, i, j = 1, \ldots, n$

9.23 $$\langle g_1;\ldots;g_n|f_1;\ldots;f_n\rangle = \sum_{\pi}\varepsilon(\pi)\prod_{i=1}^{n}\langle g_i|f_{\pi(i)}\rangle$$

Proof: By linearity, 9.9, and definition 9.15

Corollary: For fermions $\forall\ |g_i\rangle, |f_j\rangle \in \mathbb{H}$, $i, j = 1, \ldots, n$

9.24 $$\langle g_1;\ldots;g_n | f_1;\ldots;f_n \rangle = \det\langle g_i | f_j \rangle$$

Proof: This is the definition of the determinant.

Theorem: $\forall n \in \mathbb{N}$, such that $0 < n$, $(\mathscr{F} \cap \mathbb{H}^n) \subset (\mathscr{F} \cap \mathbb{H}^{n+1})$ is an isomorphic embedding under the mapping $\mathbb{H}^n \to \mathbb{H}^{n+1}$ given by

9.25 $$\forall\, \boldsymbol{x}^i \in \mathrm{N},\ i = 1, \ldots, n \quad |\boldsymbol{x}^1;\ldots;\boldsymbol{x}^n\rangle \to |\,\rangle|\boldsymbol{x}^1;\ldots;\boldsymbol{x}^n\rangle = |;\boldsymbol{x}^1;\ldots;\boldsymbol{x}^n\rangle$$

Proof: By 9.22 and 8.2

$$\begin{aligned}\langle\, ;\boldsymbol{y}^1;\ldots;\boldsymbol{y}^n |\, ;\boldsymbol{x}^1;\ldots;\boldsymbol{x}^n\rangle &= \sum_{\pi \neq 1} \varepsilon(\pi)\langle\,|\,\rangle \prod_{i=2}^{n+1} \langle \boldsymbol{y}^i | \boldsymbol{x}^{\pi(i)}\rangle \\ &= \langle \boldsymbol{y}^1;\ldots;\boldsymbol{y}^n | \boldsymbol{x}^1;\ldots;\boldsymbol{x}^n\rangle\end{aligned}$$

by 8.1, and using 9.22 again.

10 Annihilation Operators

In an interaction particles may be created, as described by creation operators, and particles may change state or be destroyed. The destruction of a particle in an interaction is described by the action of an annihilation operator. A change of state of a particle can be described as the annihilation of one state and the creation of another, so a complete description of any process in interaction can be achieved through combinations of creation and annihilation operators. Annihilation operators incorporate the idea that it is impossible to tell which particle of a given type has been destroyed in the interaction. They are defined by their action on a basis of $\mathscr{H}$, and their relationship to creation operators will be determined. The use of bras to denote annihilation operators is justified by the obvious homomorphism defined below in 10.2 with $n = 1$.

Definition: $\forall |\boldsymbol{x}\rangle \in \mathbb{H}_0^1$ the annihilation operator $\langle \boldsymbol{x}|{:}\mathbb{H}^n \to \mathbb{H}^{n-1}$ $\langle \boldsymbol{x}|{:}|f\rangle \to \langle \boldsymbol{x}|f\rangle \in \mathbb{H}^{n-1}$ is given by $\forall\, \boldsymbol{x}^i \in \mathrm{N}$, $i = 1, \ldots, n$

10.1 $$\langle \boldsymbol{x}||\rangle = \langle \boldsymbol{x}|\rangle$$

10.2 $$\langle \boldsymbol{x}|(|\boldsymbol{x}^1\rangle, \ldots, |\boldsymbol{x}^n\rangle) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \kappa^i \langle \boldsymbol{x}|\boldsymbol{x}^i\rangle(|\boldsymbol{x}^1\rangle, \ldots, |\boldsymbol{x}^{i-1}\rangle, |\boldsymbol{x}^{i+1}\rangle, \ldots, |\boldsymbol{x}^n\rangle)$$

The normalisation in 10.2 is determined by observing that when all $\boldsymbol{x}, \boldsymbol{x}^i$ are distinct, the right hand side is the sum of n orthogonal vectors, normalised to χ^{3n} by 8.6. $\kappa = 1$ for bosons and $\kappa = -1$ for fermions, and is determined by considering the result of the annihilation operator on a state of one particle in $\mathbb{H}^1 \subset \mathbb{H}^n \cap \mathscr{F}$, which is identical for all values of n under the isomorphic embedding of 9.25. The annihilation operator for any ket is defined by linearity

10.3 $$\forall\ |f\rangle \in \mathbb{H} \quad \langle f|{:}\mathscr{F} \to \mathscr{F} \text{ is given by } \langle f| = \sum_{\boldsymbol{x} \in \mathrm{N}} \langle f|\boldsymbol{x}\rangle\langle \boldsymbol{x}|$$

Lemma: $\forall |\boldsymbol{x}\rangle, |\boldsymbol{x}^1\rangle, |\boldsymbol{x}^2\rangle \in \mathbb{H}_0^1$

10.4 $$\langle \boldsymbol{x}|(|\boldsymbol{x}^1\rangle, |\boldsymbol{x}^2\rangle) = \frac{1}{\sqrt{2}}\langle \boldsymbol{x}|\boldsymbol{x}^1\rangle|\boldsymbol{x}^2\rangle + \kappa\langle \boldsymbol{x}|\boldsymbol{x}^2\rangle|\boldsymbol{x}^1\rangle$$

Proof: This is 10.2 with $n = 2$

Theorem: $\forall |\boldsymbol{y}\rangle, |\boldsymbol{x}^i\rangle \in \mathbb{H}_0^1, i = 1, \ldots, n$

10.5 $$\langle \boldsymbol{y}| \boldsymbol{x}^1;\ldots;\boldsymbol{x}^n\rangle = \sum_{i=1}^{n} \kappa^i \langle \boldsymbol{y}|\boldsymbol{x}^i\rangle |\boldsymbol{x}^1;\ldots;\boldsymbol{x}^{i-1};\boldsymbol{x}^{i+1};\ldots;\boldsymbol{x}^n\rangle$$

Proof: By 9.16

$$\begin{aligned}\langle \boldsymbol{y}||\boldsymbol{x}^1;\ldots;\boldsymbol{x}^n\rangle &= \langle \boldsymbol{y}|\frac{1}{\sqrt{n!}}\sum_{\pi}\varepsilon(\pi)(|\boldsymbol{x}^{\pi(1)}\rangle, \ldots, |\boldsymbol{x}^{\pi(n)}\rangle)\\ &= \frac{1}{\sqrt{n}}\frac{1}{\sqrt{n!}}\sum_{i=1}^{n}\kappa^i\langle \boldsymbol{y}|\boldsymbol{x}^i\rangle n\sum_{\pi\neq i}\varepsilon(\pi)(|\boldsymbol{x}^{\pi(1)}\rangle, \ldots, |\boldsymbol{x}^{\pi(n)}\rangle)\end{aligned}$$

by 10.2, since for each value of $i \in \{1, \ldots, n\}$ there are n permutations π which are identical apart from the position of i. 10.5 follows by applying 9.16 again.

Theorem: $\forall |\boldsymbol{x}^i\rangle, |\boldsymbol{y}^i\rangle \in \mathbb{H}_0^1, i = 1, \ldots, n$

10.6 $$\langle \boldsymbol{y}^n|\ldots\langle \boldsymbol{y}^1||\boldsymbol{x}^1;\ldots;\boldsymbol{x}^n\rangle = \langle \boldsymbol{y}^1;\ldots;\boldsymbol{y}^n|\boldsymbol{x}^1;\ldots;\boldsymbol{x}^n\rangle$$

Proof: From 9.1

$$\begin{aligned}\langle \boldsymbol{y}^1||\boldsymbol{x}^1;\boldsymbol{x}^2\rangle &= \tfrac{1}{2}\sqrt{2}\langle \boldsymbol{y}^1|[(|\boldsymbol{x}^1\rangle, |\boldsymbol{x}^2\rangle) + \kappa(|\boldsymbol{x}^2\rangle, |\boldsymbol{x}^1\rangle)]\\ &= \tfrac{1}{2}((1+\kappa^2)\langle \boldsymbol{y}^1|\boldsymbol{x}^1\rangle|\boldsymbol{x}^2\rangle + 2\kappa\langle \boldsymbol{y}^1|\boldsymbol{x}^2\rangle|\boldsymbol{x}^1\rangle)\end{aligned}$$

by applying 10.4. Then

$$\begin{aligned}\langle \boldsymbol{y}^2|\langle \boldsymbol{y}^1||\boldsymbol{x}^1;\boldsymbol{x}^2\rangle &= \tfrac{1}{2}((1+\kappa^2)\langle \boldsymbol{y}^1|\boldsymbol{x}^1\rangle\langle \boldsymbol{y}^2|\boldsymbol{x}^2\rangle + 2\kappa\langle \boldsymbol{y}^1|\boldsymbol{x}^2\rangle\langle \boldsymbol{y}^2|\boldsymbol{x}^1\rangle)\\ &= \langle \boldsymbol{y}^1|\boldsymbol{x}^1\rangle\langle \boldsymbol{y}^2|\boldsymbol{x}^2\rangle + \kappa\langle \boldsymbol{y}^1|\boldsymbol{x}^2\rangle\langle \boldsymbol{y}^2|\boldsymbol{x}^1\rangle\\ &= \langle \boldsymbol{y}^1;\boldsymbol{y}^2|\boldsymbol{x}^1;\boldsymbol{x}^2\rangle\end{aligned}$$

by 9.22. So 10.6 holds for $n = 2$. Now suppose that 10.6 holds for $n < m \in \mathbb{N}$ and apply $\langle \boldsymbol{y}^m|\ldots\langle \boldsymbol{y}^2|$ to 10.5

$$\begin{aligned}\langle \boldsymbol{y}^m|\ldots\langle \boldsymbol{y}^2|\langle \boldsymbol{y}^1|\boldsymbol{x}^1;\ldots;\boldsymbol{x}^m\rangle &= \sum_{i=1}^{m}\kappa^i\langle \boldsymbol{y}^1|\boldsymbol{x}^i\rangle\langle \boldsymbol{y}^2;\ldots;\boldsymbol{y}^m|\boldsymbol{x}^1;\ldots;\boldsymbol{x}^{i-1};\boldsymbol{x}^{i+1};\ldots;\boldsymbol{x}^m\rangle\\ &= \sum_{i=1}^{m}\kappa^i\langle \boldsymbol{y}^1|\boldsymbol{x}^i\rangle\sum_{\pi\neq i}\varepsilon(\pi)\prod_{i=2}^{m}\langle \boldsymbol{y}^i|\boldsymbol{x}^{\pi(i)}\rangle && \text{by 9.22}\\ &= \sum_{\pi}\varepsilon(\pi)\prod_{i=1}^{m}\langle \boldsymbol{y}^i|\boldsymbol{x}^{\pi(i)}\rangle && \text{since all } m \text{ terms are identical}\\ &= \langle \boldsymbol{y}^1;\ldots;\boldsymbol{y}^m|\boldsymbol{x}^1;\ldots;\boldsymbol{x}^m\rangle\end{aligned}$$

by 9.22. So 10.6 holds for $\forall n \in \mathbb{N}$ by induction.

Corollary: $\forall |\boldsymbol{x}^i\rangle \in \mathbb{H}_0^1, i = 1, \ldots, n \;\; \forall |f\rangle \in \mathbb{H}^n \cap \mathscr{F}$

$$\langle \boldsymbol{x}^1;\ldots;\boldsymbol{x}^n|f\rangle = \langle \boldsymbol{x}^n|\ldots\langle \boldsymbol{x}^1||f\rangle$$

Proof: Immediate from 10.6, by linearity. Hence, it is consistent to define:

Definition: $\forall |\boldsymbol{x}^i\rangle \in \mathbb{H}_0^1, i = 1, \ldots, n$ the annihilation operator $\langle \boldsymbol{x}^1;\ldots;\boldsymbol{x}^n|: \mathscr{H} \to \mathscr{H}$ is given by

10.7 $$\langle \boldsymbol{x}^1;\ldots;\boldsymbol{x}^n| = \langle \boldsymbol{x}^n|\ldots\langle \boldsymbol{x}^1|$$

Definition: On a complex vector space, $\mathscr{V}$, with a hermitian form, the hermitian conjugate $\phi^\dagger : \mathscr{V} \to \mathscr{V}$ of the linear operator $\phi : \mathscr{V} \to \mathscr{V}$ is defined such that $\forall f, g \in \mathscr{V}.\ (\phi^\dagger f, g) = (f, \phi g)$. It is routine to show that $\phi^\dagger$ is a linear operator.

Theorem: $\forall |\boldsymbol{x}^i\rangle \in \mathbb{H}_0^1, i = 1, \ldots, n$ the creation operator $|\boldsymbol{x}^1;\ldots;\boldsymbol{x}^n\rangle : \mathscr{F} \to \mathscr{F}$ is the hermitian conjugate of the annihilation operator, $\langle \boldsymbol{x}^1;\ldots;\boldsymbol{x}^n| : \mathscr{F} \to \mathscr{F}$.

10.8 $$\langle \boldsymbol{x}^1;\ldots;\boldsymbol{x}^n| = |\boldsymbol{x}^1;\ldots;\boldsymbol{x}^n\rangle^\dagger$$

Proof: From the definition, $\forall \boldsymbol{x}^i, \boldsymbol{y}^j \in \mathrm{N}, i = 1, \ldots, n, j = 1, \ldots, m\ \forall |f\rangle \in \mathscr{F}$,

$$\begin{aligned}\langle \boldsymbol{y}^1;\ldots;\boldsymbol{y}^n|\langle \boldsymbol{x}^1;\ldots;\boldsymbol{x}^n|^\dagger|f\rangle &= \langle \boldsymbol{y}^1;\ldots;\boldsymbol{y}^n|\langle \boldsymbol{x}^1;\ldots;\boldsymbol{x}^n||f\rangle \\ &= \langle \boldsymbol{y}^n|\ldots\langle \boldsymbol{y}^1|\langle \boldsymbol{x}^n|\ldots\langle \boldsymbol{x}^1||f\rangle \\ &= \langle \boldsymbol{x}^1;\ldots;\boldsymbol{x}^n;\boldsymbol{y}^1;\ldots;\boldsymbol{y}^n||f\rangle\end{aligned}$$

by applying 10.7 three times. Thus $\langle \boldsymbol{x}^1;\ldots;\boldsymbol{x}^n|^\dagger$ is the map

$$\langle \boldsymbol{x}^1;\ldots;\boldsymbol{x}^n|^\dagger : |\boldsymbol{y}^1;\ldots;\boldsymbol{y}^n\rangle \to |\boldsymbol{x}^1;\ldots;\boldsymbol{x}^n;\boldsymbol{y}^1;\ldots;\boldsymbol{y}^n\rangle$$

which demonstrates 10.8.

Corollary: $\forall |g\rangle, |f\rangle \in \mathbb{H}$ the annihilation operators obey the (anti)commutation relations.

10.9 $$[\langle g|, \langle f|]_\pm = 0$$

Proof: Straightforward from, 9.17, the (anti)commutation relations for creation operators.

Theorem: $\forall |g\rangle, |f\rangle \in \mathbb{H}$ the creation operators and annihilation operators obey the (anti)commutation relations

10.10 $$[\langle g|, |f\rangle]_\pm = \langle g|f\rangle$$

Proof: By 10.5, $\forall |\boldsymbol{y}\rangle, |\boldsymbol{x}^i\rangle \in \mathbb{H}_0^1, i = 1, \ldots, n$

$$\begin{aligned}\langle \boldsymbol{y}|\boldsymbol{x};\boldsymbol{x}^1;\ldots;\boldsymbol{x}^n\rangle &= \sum_{i=1}^{n} \kappa^{i+1}\langle \boldsymbol{y}|\boldsymbol{x}^i\rangle|\boldsymbol{x};\boldsymbol{x}^1;\ldots;\boldsymbol{x}^{i-1};\boldsymbol{x}^{i+1};\ldots;\boldsymbol{x}^n\rangle \\ &\quad + \langle \boldsymbol{y}|\boldsymbol{x}\rangle|\boldsymbol{x}^1;\ldots;\boldsymbol{x}^n\rangle\end{aligned}$$

Using 10.5 again

$$\langle \boldsymbol{y}||\boldsymbol{x}\rangle|\boldsymbol{x}^1;\ldots;\boldsymbol{x}^n\rangle = \kappa|\boldsymbol{x}\rangle\langle \boldsymbol{y}|\boldsymbol{x}^1;\ldots;\boldsymbol{x}^n\rangle + \langle \boldsymbol{y}|\boldsymbol{x}\rangle|\boldsymbol{x}^1;\ldots;\boldsymbol{x}^n\rangle$$

Therefore

10.11 $$[\langle \boldsymbol{y}|, |\boldsymbol{x}\rangle]_\pm = \langle \boldsymbol{y}|\boldsymbol{x}\rangle$$

10.10 follows from 9.9 and 10.3.

Corollary: $\forall |f\rangle \in \mathbb{H}$ the annihilation operator obeys the (anti)commutation relation

10.12 $$[|\rangle, \langle f|]_\pm = 0$$

11 Classical Correspondence

Real measurements do not achieve an accuracy in the order of chronons. In a measurement of position, the ket labelling the initial state $|f\rangle$ of the apparatus is changed into a ket labelling a position in *X*, a region of space of size determined by the measuring apparatus. The operator effecting the change is

11.1 $$Z(X) = \sum_{\boldsymbol{x} \in X} \frac{1}{\chi^3} |\boldsymbol{x}\rangle\langle\boldsymbol{x}|$$

as is shown by direct application

11.2 $$Z(X)|f\rangle = \sum_{\boldsymbol{x} \in X} \frac{1}{\chi^3} |\boldsymbol{x}\rangle\langle\boldsymbol{x}|f\rangle$$

since the resulting state is a weighted logical or between positions in *X*. Simultaneously, the bra labelling the initial state $\langle f|$ of the particle is changed into a bra also labelling a position in *X*.
The same operator, *Z*, causes the change

11.3 $$\langle f|Z(X) = \sum_{\boldsymbol{x} \in X} \frac{1}{\chi^3} \langle f|\boldsymbol{x}\rangle\langle\boldsymbol{x}||$$

Applying *Z* a second time to 11.2 gives

$$Z(X)Z(X)|f\rangle = \sum_{\boldsymbol{y} \in X} \frac{1}{\chi^3} |\boldsymbol{y}\rangle\langle\boldsymbol{y}| \sum_{\boldsymbol{x} \in X} \frac{1}{\chi^3} |\boldsymbol{x}\rangle\langle\boldsymbol{x}|f\rangle$$

$$= \sum_{\boldsymbol{y} \in X} \frac{1}{\chi^3} |\boldsymbol{y}\rangle\langle\boldsymbol{y}|f\rangle \qquad \text{by 5.1}$$

So $Z(X)$, is a projection operator [21], i.e.

11.4 $$Z(X)Z(X) = Z(X) \qquad \text{by 11.1}$$

reflecting the observation that a second measurement of a quantity gives the same result as the first. By simultaneously applying both changes, 11.2 and 11.3, to the initial states of apparatus $|f\rangle$, and particle $\langle f|$ described in the certainty relation $\langle f|f\rangle = 1$ obtain

11.5 $$\langle f|Z(X)Z(X)|f\rangle = \sum_{\boldsymbol{x} \in X} \frac{1}{\chi^3} \langle f|\boldsymbol{x}\rangle\langle\boldsymbol{x}|f\rangle$$

By 4.3, this is the sum of the probabilities that the particle is found at each individual position, $x \in X$. In other words it is the probability that a measurement of position finds the particle in the region *X*. In the case that *X* contains only the point ***x***, $X = \{\boldsymbol{x}\}$, 11.2 becomes

11.6 $$Z(\boldsymbol{x})|f\rangle = \frac{1}{\chi^3} |\boldsymbol{x}\rangle\langle\boldsymbol{x}|f\rangle$$

Thus, quantum position, $\langle\boldsymbol{x}|f\rangle$, can be reinterpreted as the magnitude of the projection from the state $|f\rangle$ of the apparatus into the state $|\boldsymbol{x}\rangle$, i.e. the component of $|f\rangle$ on the basis ket $|\boldsymbol{x}\rangle$. Similarly $\langle f|\boldsymbol{x}\rangle$ is the magnitude of the projection from the state $\langle f|$ of the particle into the state $\langle\boldsymbol{x}|$. 11.1 now reduces to

11.7 $$Z(X) = \sum_{\boldsymbol{x} \in X} Z(\boldsymbol{x})\langle\boldsymbol{x}|$$

According to $\mathbb{L}6$, $Z(X)$, is not simply a mathematical device to produce a result; it actually summarises the physical processes taking place in the interactions involved in a measurement of position. Thus if a measurement of position performed on the state $|f\rangle$ has resulted in a position in *X*, $Z(X)$ has, in effect, been applied to $|f\rangle$. $\mathbb{L}6$ asserts that $Z(X)$ is generated by a combination of actual particle interactions.

Classical probability theory describes situations in which every parameter exists, but some are not known. Probabilistic results come from different values taken by unknown parameters. We have a similar situation here. There are no relationships between particles apart from those generated by interactions. An experiment can be described by a large configuration of particles incorporating the measuring apparatus as well as the process being measured. The configuration of particles has been largely determined by setting up the experimental apparatus, but the precise pattern of interactions is unknown. It is clearly impossible to determine every detail of the configuration, since each detail would require an additional measurement, which would mean a larger configuration of particles with new unknown parts. Therefore there is a residual level of uncertainty, which can never be removed by experiment. Under $\mathbb{L}6$, quantum space-time generates a classical probability in which the unknowns lie in the configuration of interacting particles.

The interpretation of 11.2 is that the probability that the interactions combine to $Z(\boldsymbol{x})$, is

11.8 $$\langle f|Z(\boldsymbol{x})Z(\boldsymbol{x})|f\rangle = \langle f|Z(\boldsymbol{x})|f\rangle = \frac{1}{\chi^3}\langle f|\boldsymbol{x}\rangle\langle\boldsymbol{x}|f\rangle = \frac{1}{\chi^3}|\langle\boldsymbol{x}|f\rangle|^2$$

Thus, 4.3 can be understood as a classical probability function, where the variable, $\boldsymbol{x}$, runs over the set of projection operators,

11.9 $$Z(\boldsymbol{x}) = \frac{1}{\chi^3}|\boldsymbol{x}\rangle\langle\boldsymbol{x}|$$

such that each $Z(\boldsymbol{x})$ is generated by an unknown configuration of particle interactions in measurement. Thus classical space-time is interpreted as the overall effect of operators describing particle interactions combining into operators for measurement of time and distance.

In general, measurements generate numerical values and are repeated many times over from the same starting state. Then the average value of the result is taken. Expectation is the term used in statistics for the prediction of an average value. Under the laws of statistics, the more repetitions, the closer the average value will be to the expectation of the measurement. If $O(\boldsymbol{x})$ is a real valued function of position, $\boldsymbol{x} \in \mathrm{N}$, with probability function $|\langle f|\boldsymbol{x}\rangle|^2$, then, by definition, the expectation of $O(\boldsymbol{x})$ given the state $|f\rangle \in \mathbb{H}$ is

11.10 $$\langle O\rangle = \sum_{\boldsymbol{x}\in X}\frac{1}{\chi^3}\langle f|\boldsymbol{x}\rangle O(\boldsymbol{x})\langle\boldsymbol{x}|f\rangle$$

So if we define an operator on $\mathscr{F}$ by the formula

11.11 $$O = \sum_{\boldsymbol{x}\in X}\frac{1}{\chi^3}|\boldsymbol{x}\rangle O(\boldsymbol{x})\langle\boldsymbol{x}|$$

then the expectation of O given the initial state $|f\rangle \in \mathbb{H}$ is

11.12 $$\langle O\rangle = \langle f|O|f\rangle \qquad \text{by 11.10 and 11.11}$$

By 10.5 O is additive for independent multiparticle states, so 11.10 applies also to the expectation for all $|f\rangle \in \mathscr{F}$. O is hermitian, so there is a particular class of kets, called eigenkets, such that if $|f\rangle$ is an eigenket, then $\exists r \in \mathbb{R}$ known as the eigenvalue associated with $|f\rangle$ such that

11.13 $$O|f\rangle = r|f\rangle.$$

Then the state is known as an eigenstate. An eigenstate is described by a quantum logical truth function which represents certainty at the eigenvalue.

It is implicit in $\mathbb{L}6$ that all observable quantities are composed of interaction operators. Then the existence of an observable quantity depends not on whether an observation takes place, but on the configuration of matter. If, in the description of a physical process, the interaction operators combine to generate an observable operator, then the corresponding observable quantity exists, independent of observation or measurement. Then the physical state has been generated by a combination of interactions given by a projection operator, and is itself labelled by an eigenket of the observable. The value of the observable quantity is given by the corresponding eigenvalue, independent of observation or measurement,

11.14 $$\langle O \rangle = \langle f | O | f \rangle = \langle f | r | f \rangle = r \langle f | f \rangle = r$$

We know from experiment that measurements generate definite results, and thereby provide definite categorisations of states by means of a kets. This is equivalent to the application of a projection operator. In a statistical analysis of a large number of particles, each result labels a physical process described by a combination of operators equivalent to a projection operator. Under the identification of addition with many valued logical OR the expectation of all the results is a hermitian operator equal to a weighted sum over a family of projection operators. Thus, any measured value, such as the position of an object, is not a inherent property of space, or even a property of the object itself, but rather a value arising from the relationships of the particles in the object to other matter in the universe. Classical laws, such as our perception of three dimensional space, are derived from the statistical analysis of the behaviour of large numbers of particles.

12 Discrete Wave Mechanics

The construction of $\mathscr{F}$ requires no physics beyond the knowledge that we can measure the position of individual particles, and we can measure the relative frequency of each result of a repeated measurement. $\mathscr{F}$ is simply a labelling system for state. The description of physical processes requires a law describing the time evolution of states in the labelling system. Let $\mathrm{T} \subset \mathbb{N}$ be a finite discrete time interval such that any particle under study certainly remains in N for $x_0 \in \mathrm{T}$. Without loss of generality let $\mathrm{T} = [0, T)$.
Definition: An interaction at time t is described by an operator, $I(t){:}\mathscr{F} \to \mathscr{F}$. For definiteness we may take

12.1 $$\forall \boldsymbol{x}^i \in \mathrm{N}, \forall n \in \mathbb{N}, \langle \boldsymbol{x}^1 ;\ldots; \boldsymbol{x}^n | I | \boldsymbol{x}^1 ;\ldots; \boldsymbol{x}^n \rangle = 0$$

since otherwise there would be a component corresponding to the absence of interaction.

At each time t, either no interaction takes place and the state $|f\rangle \in \mathscr{F}$ is unchanged, or an interaction, I, takes place. By the identification of the operations of vector space with weighted OR between uncertain possibilities, the possibility of an interaction at time t is described by the map $\mathscr{F} \to \mathscr{F}$

$$|f\rangle \to \mu\left(1 - i\frac{I(t)}{\chi}\right)|f\rangle$$

where μ is a scalar value chosen to preserve the norm, as required by the probability interpretation. Thus the law of evolution of the ket in one chronon from time t to time $t+1$ is

12.2 $$|f\rangle_{t+1} = \mu\left(1 - i\frac{I(t)}{\chi}\right)|f\rangle_t$$

In the absence of interaction, the ket for a particle at rest does not change (such states exist as a particle is always at rest in its own reference frame). So $\forall |f\rangle \in \mathbb{H} \quad \exists \lambda \in \mathbb{C}$ such that 12.2 reduces to

12.3 $$|f\rangle_{t+1} = \lambda |f\rangle_t$$

Preservation of the norm implies that $\exists m \in \mathbb{R}$ such that $\lambda = e^{im}$, so that

12.4 $$|f\rangle_{t+1} = e^{im}|f\rangle_t$$

Then 12.3 is a geometric progression with solution

12.5 $$|f\rangle_t = e^{imt}|f\rangle_0$$

Definition: *m* is the bare mass of a particle.

It will be found that *m* can be identified with the classical concept of mass. By 5.1, the solution for a particle in its own reference frame is

12.6 $$\langle t, \boldsymbol{x}|\boldsymbol{0}\rangle = \chi^3 e^{imt}\delta_{\boldsymbol{x0}}$$

The quantity time used in physics is simply a number read from a clock. A clock is simply a repeating process and a counter to record the number of repetitions of the process. It does not affect the behaviour of a clock whether or not anyone reads the counter, and it does not change physical law if the counter does not work, or even if it does not exist. By $\mathbb{L}3$, if there is a repeating process, the laws of physics are always the same with regard to it. 12.2 describes a repeating process, and thereby defines a clock without a counter for each particle, and generates a time-scale of chronons associated with the particle. There may be a different chronon associated with each fundamental particle.

According $\mathbb{L}3$, the laws of physics are the same for the particle as for the matter in the macroscopic clock. So, applying $\mathbb{L}3$ to the analysis of radar, justified by $\mathbb{L}5$, when photons are emitted from the particle and photons return, basic geometrical relationships are established, as is a reference frame based on the particle's clock, called the particle's reference frame. The origin of the particle's reference frame is the quantum position of the particle, which is many valued in a macroscopic reference frame, and we refer to quantum reference frames

Definition: For any single particle state, $|f\rangle$, normalised so that $\langle f|f\rangle = 1$ the quantum transformation $Q(f)$ is given by

12.7 $$Q(f) = \int_{\mathrm{M}} d^3\boldsymbol{p}|\boldsymbol{p}\rangle\langle\boldsymbol{p}|f\rangle\langle\boldsymbol{p}|$$

Then

12.8 $$Q(f)|\boldsymbol{x}\rangle = \int_{\mathrm{M}} d^3\boldsymbol{p}|\boldsymbol{p}\rangle\langle\boldsymbol{p}|f\rangle\langle\boldsymbol{p}|\boldsymbol{x}\rangle = \left(\frac{\chi}{2\pi}\right)^{\frac{3}{2}}\int_{\mathrm{M}} d^3\boldsymbol{p}|\boldsymbol{p}\rangle e^{i\boldsymbol{x}\cdot\boldsymbol{p}}\langle\boldsymbol{p}|f\rangle$$

Thus $Q(f)|\boldsymbol{0}\rangle = |f\rangle$ and $Q(f)$ transforms the particle's reference frame to a quantum reference frame, as it appears in a macroscopic frame. *Q* is not unitary, since information is lost in a quantum transform:

12.9 $$Q^\dagger(f)|f\rangle = \int_{\mathrm{M}} d^3\boldsymbol{p}|\boldsymbol{p}\rangle\langle f|\boldsymbol{p}\rangle\langle\boldsymbol{p}|f\rangle \neq |\boldsymbol{0}\rangle$$

But for a clock with a certain position $\boldsymbol{z} \in \mathrm{N}$, $Q(\boldsymbol{z})$ is a space translation

12.10 $$Q(\boldsymbol{z}) = \int_{\mathrm{M}} d^3\boldsymbol{p}|\boldsymbol{p}\rangle\langle\boldsymbol{p}|\boldsymbol{z}\rangle\langle\boldsymbol{p}| = \left(\frac{\chi}{2\pi}\right)^{\frac{3}{2}}\int_{\mathrm{M}} d^3\boldsymbol{p}|\boldsymbol{p}\rangle e^{i\boldsymbol{z}\cdot\boldsymbol{p}}\langle\boldsymbol{p}|$$

since $\langle\boldsymbol{x}|Q(\boldsymbol{z})|f\rangle = \langle\boldsymbol{x}-\boldsymbol{z}|f\rangle$ by 7.13.

13 Continuous Wave Mechanics

From 7.6, at any time $t \in \mathrm{T}$

13.1 $$\langle \boldsymbol{x} \,|\, f \rangle = \left(\frac{\chi}{2\pi}\right)^{\frac{3}{2}} \int_{\mathrm{M}} d^3\boldsymbol{p} \, \langle \boldsymbol{p} | f \rangle \, e^{-i\boldsymbol{x}\cdot\boldsymbol{p}}$$

Although $\langle \boldsymbol{x} \,|\, f \rangle$ is, by definition, discrete, on a macroscopic time-scale it appears continuous. 13.1 can be embedded in continuous function $f{:}\mathbb{R}^3 \to \mathbb{C}$ given by

13.2 $$f(\boldsymbol{x}) = \left(\frac{\chi}{2\pi}\right)^{\frac{3}{2}} \int_{\mathrm{M}} d^3\boldsymbol{p} \, \langle \boldsymbol{p} | f \rangle \, e^{-i\boldsymbol{x}\cdot\boldsymbol{p}}$$

Similarly 12.6 can be embedded into a continuous function of time $f{:}\mathbb{R} \to \mathbb{C}$ given by

13.3 $$f(t) = \chi^3 e^{imt} \delta_{\boldsymbol{x}\boldsymbol{0}}$$

We seek a continuous function, $f{:}\mathbb{R}^4 \to \mathbb{C}$, called the wave function, such that, if the particle certainly remains in N for time interval, $\mathrm{T} \subset \mathbb{N}$, then $\langle x \,|\, f \rangle$ can be embedded into $f(x)$[1]

13.4 $$\forall \boldsymbol{x} \in \mathrm{N}, \forall t \in \mathrm{T} \quad \langle x | f \rangle = \langle t, \boldsymbol{x} \,|\, f \rangle = f(t, \boldsymbol{x}) = f(x)$$

Physical law will be expressed in terms of creation and annihilation operators, which are homomorphic to states, and must be Lorentz invariant to satisfy $\mathbb{L}3$. Clearly Lorentz transformation cannot be applied directly to functions of a discrete co-ordinate system. But it can be applied to the wave function. Then 13.4 defines quantum position, and hence a ket, by the restriction of the wave function to the transformed co-ordinate system, N at integer time. For any ket, there is a unique momentum space function defined by 7.4, and a unique wave function defined by 13.2. So there is a homomorphism between $\mathbb{H}$ and the vector space of wave functions with the hermitian product defined by 5.5

13.5 $$\langle g | f \rangle = \sum_{\boldsymbol{x} \in \mathrm{N}} \frac{1}{\chi^3} \overline{g(\boldsymbol{x})} \, f(\boldsymbol{x})$$

Wave functions are not restricted to $\mathscr{L}^2$, and 13.5 is not the hermitian product of Hilbert space, but by the definition of convergence of an integral, it is approximated by the hermitian product whenever f and g are in $\mathscr{L}^2$ and χ can be regarded as small.

Invariance under Lorentz transformation requires that the law of time evolution has a Lorentz invariant form when expressed in terms of wave functions. The law for the time evolution of the wave function for a stationary particle is given by differentiating 13.3 with respect to time

13.6 $$-i\partial_0 f = mf$$

Then 12.4 is obtained by integrating 13.6 over one chronon. Thus, in the restriction to integer values, 13.6 is identical to 12.4, the difference equation for a stationary non-interacting particle. It is therefore an expression of the same relationship or law. As an equation of the wave function, the right hand side of 13.6 is a scalar, whereas the left hand side is the time component of a vector whose space component is zero. So 13.6 is not manifestly covariant. For a covariant equation which reduces to 13.6 for a particle in its own reference frame, we take a scalar product involving the vector derivative, ∂, and the wave function

13.7 $$-i\partial \cdot \Gamma f = mf$$

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1. Gravity will be considered in another paper, but it is interesting to observe that the embedding is not dependent on the metric, and can be in a four dimensional differentiable manifold.

Then the time evolution of quantum position in any reference frame is the restriction of the solution of 13.7 to N at time $t \in \mathrm{T}$. As discovered by Dirac [22], there is no invariant equation in the form of 13.7 for scalar *f* and the theory breaks down. To rectify the problem a spin index is added to N

13.8 $$\mathrm{N}_S = \mathrm{N} \otimes S \qquad \text{for } v \in \mathbb{N}$$

where S is a finite set of indices. The constructions of the vector spaces, $\mathbb{H}$, $\mathscr{H}$ and $\mathscr{F}$, go through as before, but when we wish to make the spin index explicit we write

13.9 $$|\boldsymbol{x}\rangle = |\boldsymbol{x}, \alpha\rangle = |\boldsymbol{x}\rangle_\alpha$$

normalised by 5.1 so that

13.10 $$\forall(\boldsymbol{x}, \alpha), (\boldsymbol{y}, \beta) \in \mathrm{N}_S \quad \langle \boldsymbol{x}, \alpha | \boldsymbol{y}, \beta \rangle = \langle \boldsymbol{x} | \boldsymbol{y} \rangle_{\alpha\beta} = \chi^3 \delta_{\boldsymbol{xy}} \delta_{\alpha\beta}$$

the wave function acquires a spin index

13.11 $$f(x) = f_\alpha(x) = \langle x | f \rangle_\alpha$$

and the braket becomes

13.12 $$\langle g | f \rangle = \sum_{\boldsymbol{x} \in \mathrm{N}_S} \frac{1}{\chi^3} \langle g | \boldsymbol{x} \rangle \langle \boldsymbol{x} | f \rangle = \sum_{(\mathbf{x}, \alpha) \in \mathrm{N}_S} \frac{1}{\chi^3} \overline{g_\alpha(x)} f_\alpha(x)$$

It is now possible to find a covariant equation which reduces to 13.6 in the particle's reference frame, namely the Dirac equation,

13.13 $$i\partial \cdot \gamma f(x) = mf(x)$$

Another possibility is that *f* is a vector and that 13.6 is a representation of a vector equation with $m = 0$

13.14 $$i\partial \cdot f(x) = 0$$

The norm is intended to generate physically realisable predictions of probability, and must be both invariant and positive definite. It is given by

13.15 $$\langle f | f \rangle = \sum_{(\mathbf{x}, \alpha) \in \mathrm{N}_S} \frac{1}{\chi^3} \overline{f_\alpha(x)} f_\alpha(x)$$

If *f* transforms as a space-time vector, 13.15 is only invariant if 13.10 is replaced by the definition

13.16 $$\forall(\boldsymbol{x}, \alpha), (\boldsymbol{y}, \beta) \in \mathrm{N} \quad \langle \boldsymbol{x}, \alpha | \boldsymbol{y}, \beta \rangle = \langle \boldsymbol{x} | \boldsymbol{y} \rangle_{\alpha\beta} = \chi^3 \delta_{xy} g_{\alpha\beta} = \eta(\alpha) \chi^3 \delta_{\boldsymbol{xy}} \delta_{\alpha\beta}$$

where η is given by

13.17 $\eta(0) = -1$ and $\eta(1) = \eta(2) = \eta(3) = 1$.

and *g* is the matrix

13.18 $$g_{\alpha\beta} = \eta(\alpha)\delta_{\alpha\beta}$$

We will use the summation convention for repeated spin indices, but not the convention of raising and lowering indices. The factor -1 is implicit in summing the zeroeth index for vectors, so 13.12 and 13.15 are retained. 13.16 is invariant, but not positive definite, as required by a norm. The definition of the braket in terms of probability implies that any vector particles have a positive definite norm for physical states, so only space-like polarisation can be realised physically. It will be shown in section 16, *Plane Wave States* that this is true of the solutions of 13.14.

14 Dirac Particles

Dirac found the solution to 13.13

14.1 $$f_\alpha(x) = \left(\frac{\chi}{2\pi}\right)^{\frac{3}{2}} \sum_{r=1}^{2} \int d^3\boldsymbol{p}\ F(\boldsymbol{p}, r) u_\alpha(\boldsymbol{p}, r)\ e^{-ix \cdot p}$$

where p satisfies the mass shell condition

14.2 $$p_0 = \sqrt{m^2 + \boldsymbol{p}^2} \qquad .$$

and u is a Dirac spinor, having the form

14.3 $$u(\boldsymbol{p}, r) = \sqrt{\frac{p_0 + m}{2p_0}} \begin{bmatrix} \zeta(r) \\ \dfrac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{p_0 + m} \zeta(r) \end{bmatrix} \qquad \text{for } r = 1,2$$

where ζ is a two-spinor normalised so that

14.4 $$\bar{\zeta}_\alpha(r)\zeta_\alpha(s) = \delta_{rs}$$

and $\boldsymbol{\sigma} = (\sigma_1,\sigma_2,\sigma_3)$ are the Pauli spin matrices. It is routine to show the spinor normalisation

14.5 $$\bar{u}_a(\boldsymbol{p}, r) u_\alpha(\boldsymbol{p}, s) = \delta_{rs}$$

$F(\boldsymbol{p},r)$ is the momentum space wave function given by inverting 14.1 at $x_0 = 0$

14.6 $$F(\boldsymbol{p}, r) = \left(\frac{\chi}{2\pi}\right)^{\frac{3}{2}} \sum_{(\boldsymbol{x}, \alpha) \in \mathrm{N}} \frac{1}{\chi^3} f_\alpha(0, \boldsymbol{x}) \bar{u}_\alpha(\boldsymbol{p}, r) e^{i\boldsymbol{x} \cdot \boldsymbol{p}}$$

Definition: p_0 is the energy of a state with momentum $\boldsymbol{p}$. $p = (p_0, \boldsymbol{p})$ is called energy-momentum; p_0 will later be identified with classical energy.

Definition: With the Dirac γ-matrices as defined in the literature the Dirac adjoint is

14.7 $$\hat{u} = \bar{u}\gamma^0$$

Lemma: The γ-matrices obey the relations

14.8 $$\gamma^0\gamma^0 = 1 \text{ and } \gamma^0\gamma^{\alpha\dagger}\gamma^0 = \gamma^\alpha$$

Proof: These are familiar matrix equations and the proof is left to the reader

Lemma: In this normalisation Dirac spinors obey the following relations

14.9 $$(p \cdot \gamma - m) u(\boldsymbol{p}, r) = 0 = \hat{u}(\boldsymbol{p}, r)(p \cdot \gamma - m)$$

14.10 $$\hat{u}_\alpha(\boldsymbol{p}, r) u_\alpha(\boldsymbol{p}, s) = \delta_{rs}\frac{m}{p_0}$$

14.11 $$\sum_{r=1}^{2} u_\alpha(\boldsymbol{p}, r)\hat{u}_\beta(\boldsymbol{p}, r) = \left(\frac{p \cdot \gamma + m}{2p_0}\right)_{\alpha\beta}$$

Proof: These are familiar spinor relations renormalised and the proof is left to the reader. This normalisation is consistent the definition of ket space in the reference frame of an individual observer and leads to some simplification of the formulae. Wave functions are non-physical and it is not necessary to use the invariant integral.

Theorem: Dirac particles are fermions.

Proof: The spin statistics theorem is as in the standard theory.

The treatment of the antiparticle modifies the Stückelberg-Feynman [23],[24] interpretation by considering the mass shell condition as derived from the *k*-calculus. A sign is lost in 3.9 due to the squared terms. 3.9 is derived only for time-like vectors, and, when extended to any vector, is not strictly positive and does not define a norm which could be used as a vector magnitude. But 3.10 can apply to a time-like vector pointing backwards in time. Such a vector has a negative time-like component and a natural definition of $m < 0$. So, the permissible solutions of the Dirac equation, 13.13, have positive energy, $E = p_0 > 0$, when m is positive, and negative energy when m is negative.

By repetition of 12.2, each Dirac particle carries its own clock which marks off time in chronons. $\mathbb{L}5$ allows us to analyse radar, and postulate that photons pass between Dirac particles. Through the transfer of photons geometrical relationships are set up between particles. In the absence of geometrical relationships we cannot say whether one particle's clock counts time in the same direction as another. So, as a matter of principle, following the introduction of geometry, we may find that some clocks count backwards to others. In a conventional reference frame determined from a macroscopic clock, the particle can be shown at any time on a space-time diagram as a (quantum) vector, the arrow showing the direction of the particle's clock. The set of such vectors is the time line of the particle. If a particle's clock changes direction with respect to the macroscopic clock, the time-line of the particle becomes reversed, and it appears as though a particle and an antiparticle have annihilated each other; the annihilation of a negative energy state is seen as the creation of positive energy. Similarly, if an antiparticle's clock changes direction it is seen in the macroscopic frame as the creation of a particle-antiparticle pair.

In the particle's reference frame the particle's clock always counts forwards. The general form, 13.13, is recovered by transformation to another reference frame. So, the principle of homogeneity does not only require Lorentz covariance of the Dirac equation, but also time reversal for the negative energy solutions for which the particle's clock is running backwards. Complex conjugation of quantum position reverses time while maintaining the probability relationship 4.3, and restores positive energy. To be consistent we also have to change the sign of mass, $m \to -m$. Under $\mathbb{L}3$ there is no preferred orientation in space, and space inversion restores momentum space.

Thus, given that no interaction takes place, the ket for a Dirac particle in its own reference frame evolves according to 12.5, for both $m > 0$ and $m < 0$. But when the negative energy solution is transformed to a macroscopic reference frame the Dirac equation, 13.13, becomes

14.12 $$i\partial \cdot \bar{\gamma} f(x) = -mf(x)$$

where γ is the complex conjugate, $\bar{\gamma}^0_{\alpha\beta} = \overline{\gamma_{\alpha\beta}}$. The solution to 14.12 is the wave function for the antiparticle

14.13 $$f(x) = \left(\frac{\chi}{2\pi}\right)^{\frac{3}{2}} \sum_{r=1}^{2} \int_{\mathbb{M}} d^3\boldsymbol{p}\ F(\boldsymbol{p}, r)\bar{v}(\boldsymbol{p}, r)\ e^{-ix \cdot p}$$

where p satisfies the mass shell condition, 3.9, and $\bar{v}$ is the complex conjugate of the Dirac spinor.

14.14 $$v(\boldsymbol{p}, r) = \sqrt{\frac{p_0 + m}{2p_0}} \begin{bmatrix} \dfrac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{p_0 + m}\zeta(r) \\ \zeta(r) \end{bmatrix} \qquad \text{for } r = 1,2$$

14.13 is the complex conjugate of the negative energy solution of the Dirac equation. The spinor has the normalisation

14.15 $$\bar{v}_\alpha(\boldsymbol{p}, r)v_\alpha(\boldsymbol{p}, s) = \delta_{rs}$$

$F(\boldsymbol{p},r)$ is the momentum space wave function given by

14.16 $$F(\boldsymbol{p}, r) = \left(\frac{\chi}{2\pi}\right)^{\frac{3}{2}} \sum_{(\boldsymbol{x}, \alpha) \in N} \frac{1}{\chi^3} f_\alpha(0, \boldsymbol{x}) v_\alpha(\boldsymbol{p}, r) e^{i\boldsymbol{x} \cdot \boldsymbol{p}}$$

Lemma: In this normalisation the Dirac spinors obey the following relations

14.17 $$(p \cdot \gamma + m) v(\boldsymbol{p}, r) = 0 = \hat{v}(\boldsymbol{p}, r)(p \cdot \gamma + m)$$

14.18 $$\hat{v}_\alpha(\boldsymbol{p}, r) v_\alpha(\boldsymbol{p}, s) = \delta_{rs} \frac{m}{p_0}$$

14.19 $$\sum_{r=1}^{2} v_\alpha(\boldsymbol{p}, r) \hat{v}_\beta(\boldsymbol{p}, r) = \left(\frac{p \cdot \gamma - m}{2p_0}\right)_{\alpha\beta}$$

Proof: These are familiar spinor relations renormalised and the proof is left to the reader.

15 The Photon

Theorem: The solution to 13.14 is the wave function for the photon

15.1 $$f_\alpha(x) = \left(\frac{\chi}{2\pi}\right)^{\frac{3}{2}} \int_M d^3\boldsymbol{p}\ F(\boldsymbol{p}, r) w_\alpha(\boldsymbol{p}, r)\ e^{-ix \cdot p} \quad \text{where}$$

i. $p^2 = 0$

ii. w are orthonormal vectors given by

a) time-like component: $w(\boldsymbol{p}, r) = (1, \boldsymbol{0})$

b) space-like components: for $r = 1,2,3$ $w(\boldsymbol{p}, r) = (0, \boldsymbol{w}(\boldsymbol{p}, r))$ are such that

$\boldsymbol{w}(\boldsymbol{p}, r) \cdot \boldsymbol{w}(\boldsymbol{p}, s) = \delta_{rs}$ and $\boldsymbol{w}(\boldsymbol{p}, 3) = \dfrac{\boldsymbol{p}}{p_0}$ is longitudinal, so $\boldsymbol{w}(\boldsymbol{p},1)$ and $\boldsymbol{w}(\boldsymbol{p},2)$ are transverse

iii. F is such that the photon cannot be polarised in the longitudinal and time-like spin states, i.e.

15.2 $$F(\boldsymbol{p},0) = F(\boldsymbol{p},3)$$

Proof: With the above definitions

15.3 $$p.w(\boldsymbol{p},3) = p_0 = -\, p.w(\boldsymbol{p},0) \text{ and } p.w(\boldsymbol{p},1) = p.w(\boldsymbol{p},2) = 0$$

So that differentiating 15.1

$$\begin{aligned} i\partial \cdot f(x) &= \left(\frac{\chi}{2\pi}\right)^{\frac{3}{2}} \sum_{r=0}^{3} \int_M d^3\boldsymbol{p}\ F(\boldsymbol{p}, r) p \cdot w_\alpha(\boldsymbol{p}, r)\ e^{-ix \cdot p} \\ &= \left(\frac{\chi}{2\pi}\right)^{\frac{3}{2}} \sum_{r=0}^{3} \int_M d^3\boldsymbol{p}\ p_0(F(\boldsymbol{p}, 3) - F(\boldsymbol{p}, 0)) e^{-ix \cdot p} \\ &= 0 \qquad \text{by 15.2} \end{aligned}$$

which establishes that 15.1 is the solution to 13.14

$F(\boldsymbol{p},r)$ is the momentum space wave function given by inverting 15.1 at $x_0 = 0$

15.4 $$F(\boldsymbol{p}, r) = \left(\frac{\chi}{2\pi}\right)^{\frac{3}{2}} \eta(r) \sum_{(\boldsymbol{x}, \alpha) \in N} \frac{1}{\chi^3} f_\alpha(0, \boldsymbol{x}) w_\alpha(\boldsymbol{p}, r) e^{i\boldsymbol{x} \cdot \boldsymbol{p}}$$

The photon has zero mass and 13.14 does not define a repeating process or a direction of time. It is unchanged under time reversal and is its own antiparticle.

16 Plane Wave States

Definition: $\forall x_0 \in \mathbb{T}$ plane wave states $|\boldsymbol{p}, r\rangle = \mathbb{H}$ are defined by the wave functions

16.1 $$\langle x|\boldsymbol{p}, r\rangle = \left(\frac{\chi}{2\pi}\right)^{\frac{3}{2}} u(\boldsymbol{p}, r)e^{-ix \cdot p}$$ for the Dirac particle

16.2 $$\langle x|\boldsymbol{p}, r\rangle = \left(\frac{\chi}{2\pi}\right)^{\frac{3}{2}} \bar{v}(\boldsymbol{p}, r)e^{-ix \cdot p}$$ for the antiparticle, and

16.3 $$\langle x|\boldsymbol{p}, r\rangle = \left(\frac{\chi}{2\pi}\right)^{\frac{3}{2}} w(\boldsymbol{p}, r)e^{-ix \cdot p}$$ for the photon.

Theorem: (Newton's first law) In an inertial reference frame, an elementary particle in isolation has a constant momentum space wave function. $\forall|f\rangle \in \mathbb{H}$

16.4 $$|f\rangle = \sum_r \eta(r) \int_{\mathbb{M}} d^3\boldsymbol{p}|\boldsymbol{p}, r\rangle\langle\boldsymbol{p}, r|f\rangle$$

Proof: Clearly plane waves are solutions of 13.13, 14.12 and 13.14 so they describe the evolution of states in isolation. For each of the Dirac particle, antiparticle and photon, by 5.4, $\forall|f\rangle \in \mathbb{H}$, $\forall x_0 \in \mathbb{T}$

16.5 $$\langle\boldsymbol{p}, r|f\rangle = \sum_{\boldsymbol{x} \in \mathbb{N}} \frac{1}{\chi^3}\langle\boldsymbol{p}, r|x\rangle\langle x|f\rangle$$

Substituting 16.1, 16.2 and 16.3 in 16.5, with $x_0 = 0$, and examining 14.6, 14.13 and 15.4 reveals

16.6 $$\langle\boldsymbol{p}, r|f\rangle = F(\boldsymbol{p}, r)$$ for the Dirac particles, and

16.7 $$\langle\boldsymbol{p}, r|f\rangle = \eta(r)F(\boldsymbol{p}, r)$$ for the photon.

Corollary: The time evolution of quantum position of a particle in isolation is, $\forall|f\rangle \in \mathbb{H}$

16.8 $$\langle x|f\rangle = \sum_r \eta(r) \int_{\mathbb{M}} d^3\boldsymbol{p}\langle x|\boldsymbol{p}, r\rangle\langle\boldsymbol{p}, r|f\rangle$$

where r = 0-3 for photons, and r = 1-2 for Dirac particles (η is redundant for a Dirac particle).

Proof: Substituting 16.6 and 16.1 into 14.1, 16.6 and 16.2 into 14.13, and 16.7 and 16.3 into 15.1 gives, in each case, 16.8.

Corollary: The resolution of unity

16.9 $$\sum_r \eta(r) \int_{\mathbb{M}} d^3\boldsymbol{p}|\boldsymbol{p}, r\rangle\langle\boldsymbol{p}, r| = 1$$

Proof: 16.4 is true for all $|f\rangle \in \mathbb{H}$.

Corollary: The braket has the time invariant form

16.10 $$\langle g|f\rangle = \sum_r \eta(r) \int_{\mathbb{M}} d^3\boldsymbol{p}\langle g|\boldsymbol{p}, r\rangle\langle\boldsymbol{p}, r|f\rangle$$

Proof: Immediate from 16.9

Theorem: $\langle\boldsymbol{q}, s|\boldsymbol{p}, r\rangle$ is a delta function on the test space of momentum space wave functions

16.11 $$\langle\boldsymbol{q}, s|\boldsymbol{p}, r\rangle = \eta(r)\delta_{rs}\delta(\boldsymbol{p} - \boldsymbol{q})$$

Proof: From 16.10, for plane wave $|\boldsymbol{q}, s\rangle$

$$\langle\boldsymbol{q}, s|f\rangle = \sum_r \eta(r) \int_{\mathbb{M}} d^3\boldsymbol{p}\langle\boldsymbol{q}, s|\boldsymbol{p}, r\rangle\langle\boldsymbol{p}, r|f\rangle$$

Corollary: The braket for the photon is positive definite, as required by the probability interpretation.

Proof: By 15.2 and 16.7 the time-like ($r = 0$) and longitudinal ($r = 3$) states cancel out in 16.10 and for photons as well as Dirac particles 16.10 reduces to

16.12 $$\langle g | f \rangle = \sum_{r=1}^{2} \int_{M} d^3\boldsymbol{p} \langle g | \boldsymbol{p}, r \rangle \langle \boldsymbol{p}, r | f \rangle$$

Theorem: (Gauge invariance). Let g be an arbitrary solution of $\partial^2 g = 0$. Then observable results are invariant under gauge transformation of the photon wave function given by

16.13 $$f_\alpha(x) \to f_\alpha(x) + \partial_\alpha g(x)$$

Proof: It follows from 16.12 that the braket is invariant under the addition of a (non-physical) light-like polarisation state, known as a gauge term. Let $G(\boldsymbol{p})$ be an arbitrary function of momentum. The general solution for g is

$$g = \int_{M} d^3\boldsymbol{p} e^{-ip \cdot x} G(\boldsymbol{p})$$

where $p^2 = 0$. Then

$$\partial_\alpha g = \int_{M} d^3\boldsymbol{p} p_0 (w(\boldsymbol{p}, 0) + w(\boldsymbol{p}, 3)) e^{-ip \cdot x} G(\boldsymbol{p})$$

is equivalent to a light like polarisation states, and has no effect on the braket.

$\partial_\alpha g$ is known as a gauge term, and has no physical meaning. It follows from 16.12 that light-like polarisation cannot be determined from experimental results. Although their value is hidden by the gauge term, the time-like and longitudinal polarisation states cannot be excluded, and we will see that they contribute to the electromagnetic force.

Theorem: Space-time translation by displacement z, of the co-ordinate system such that the particle remains in N, is equivalent to multiplication of the momentum space wave function by $e^{ip \cdot z}$ (c.f. 7.13).

Proof: Using 16.6 and/or 16.7 in 16.4.

16.14 $$\langle x | f \rangle = \sum_r \int_{M} d^3\boldsymbol{p} F(\boldsymbol{p}, r) \langle x | \boldsymbol{p}, r \rangle$$

Under a space-time translation, z, by 16.1, 16.2 and 16.3 we have,

16.15 $$\langle x - z | f \rangle = \sum_r \int_{M} d^3\boldsymbol{p} F(\boldsymbol{p}, r) e^{ix \cdot z} \langle x | \boldsymbol{p}, r \rangle$$

as required.

17 The Reduction of the Wave Packet

The wave function of a Dirac particle localised at $(\boldsymbol{y}, \beta) \in \mathrm{N}$ at time $y_0 \in \mathrm{T}$ is given by 14.1

$$\langle x | y \rangle_{\alpha\beta} = \left(\frac{\chi}{2\pi}\right)^{\frac{3}{2}} \sum_{r=1}^{2} \int_{M} d^3\boldsymbol{p}\ Y_\beta(\boldsymbol{p}, r) u_\alpha(\boldsymbol{p}, r)\ e^{-ix \cdot p}$$

where, by the property of the basis the momentum space function 14.6 reduces to

$$Y_\beta(\boldsymbol{p}, r) = \left(\frac{\chi}{2\pi}\right)^{\frac{3}{2}} \bar{u}_\beta(\boldsymbol{p}, r) e^{iy \cdot p}$$

Hence

17.1 $$\langle x|y\rangle_{\alpha\beta} = \left(\frac{\chi}{2\pi}\right)^3 \sum_{r=1}^{2} \int_M d^3\boldsymbol{p}\ \bar{u}_\beta(\boldsymbol{p}, r)u_\alpha(\boldsymbol{p}, r)\ e^{ip\cdot y - ix\cdot p}$$

Although there are four indices, there are only two independent spin states. Examination of 14.3 reveals that for non-relativistic values of momentum the third and fourth indices contribute to the braket with negligible amplitude and can be ignored. So u can be replaced with ζ and 17.1 reduces to

17.2 $$\text{For } \alpha, \beta = 0, 1 \quad \langle x|y\rangle_{\alpha\beta} \approx \frac{\chi^3}{8\pi^3}\delta_{\alpha\beta}\int_M d^3\boldsymbol{p}\ e^{ip\cdot y - ix\cdot p}$$

by 14.4, the orthonormality of ζ. When $x_0 = y_0$, by 7.5, 17.2 is a Kronecker delta describing an exact position at $\boldsymbol{y}$,

17.3 $$\text{For } \alpha, \beta = 0, 1, \quad \text{at } x_0 = y_0, \quad \langle \boldsymbol{x}|\boldsymbol{y}\rangle_{\alpha\beta} \approx \chi^3\delta_{\alpha\beta}\delta_{\boldsymbol{xy}}$$

Similarly, for antiparticle states

17.4 $$\text{For } \alpha, \beta = 2, 3, \quad \text{at } x_0 = y_0, \quad \langle \boldsymbol{x}|\boldsymbol{y}\rangle_{\alpha\beta} \approx \chi^3\delta_{\alpha\beta}\delta_{\boldsymbol{xy}}$$

For a photon ket $||y\rangle \in \mathbb{H}_0$ the wave function at time y_0 is given by 15.1

$$\langle x|y\rangle_{\alpha\beta} = \left(\frac{\chi}{2\pi}\right)^{\frac{3}{2}} \sum_{r=0}^{3} \int_M d^3\boldsymbol{p}\ Y_\beta(\boldsymbol{p}, r)w_\alpha(\boldsymbol{p}, r)\ e^{-ix\cdot p}$$

where, from 15.4

$$Y(\boldsymbol{p}, r) = \left(\frac{\chi}{2\pi}\right)^{\frac{3}{2}}\eta(r)w_\beta(\boldsymbol{p}, r)e^{iy\cdot p}.$$

Hence

$$\langle x|y\rangle_{\alpha\beta} = \left(\frac{\chi}{2\pi}\right)^{\frac{3}{2}} \sum_{r=0}^{3} \int_M d^3\boldsymbol{p}\ w_\beta(\boldsymbol{p}, r)w_\alpha(\boldsymbol{p}, r)\ e^{ip\cdot y - ix\cdot p}$$

So by the orthonormality of w

17.5 $$\langle x|y\rangle_{\alpha\beta} = \frac{\chi^3}{8\pi^3}g_{\alpha\beta}\int_M d^3\boldsymbol{p}\ \ e^{ip\cdot y - ix\cdot p}$$

Thus, at $x_0 = y_0$

17.6 $$\langle \boldsymbol{x}|\boldsymbol{y}\rangle_{\alpha\beta} = \chi^3 g_{\alpha\beta}\delta_{\boldsymbol{xy}}$$

17.2, and 17.5 exhibit the reduction of the wave packet. At $x_0 = y_0$ they are delta functions, describing the localisation of the particle at the point $\boldsymbol{y}$. But at $x_0 = y_0 \pm 1$ they describe a wave function spread over co-ordinate space. This apparently defies the principle that no effect should travel faster than the speed of light. The paradox is simply resolved by recognising that the wave function is not a physical phenomenon but a set of truth values in a many valued logic applied to a labelling system.

18 Interactions

The general equation governing the evolution of kets is 12.2. Under $\mathbb{L}7$, the precise form of the interaction, I, is an assumption, but some general considerations restrict the forms the interaction can take.

Lemma: Let $A: \mathcal{H} \to \mathcal{H}$ be a hermitian operator. Then

18.1 $$\forall |f\rangle \in \mathcal{H}, \langle f|A|f\rangle = 0 \Rightarrow A = 0$$

Proof: $\forall|g\rangle, |f\rangle \in \mathscr{H}$ **,** $\forall\lambda \in \mathbb{C}$

$$(\langle f| - \lambda\langle g|)A(|f\rangle - \lambda|g\rangle) = 0$$

$$\langle f|A|f\rangle + \lambda^2\langle g|A|g\rangle - \bar{\lambda}\langle g|A|f\rangle - \lambda\langle f|A|g\rangle = 0$$

$$-\bar{\lambda}\langle g|A|f\rangle - \lambda\langle f|A|g\rangle = 0 \qquad \text{under the condition of 18.1,}$$

$$\bar{\lambda}\langle g|A|f\rangle = -\lambda\overline{\langle g|A|f\rangle} \qquad \text{since } A \text{ is hermitian}$$

Then $\lambda = 1 \Rightarrow \mathscr{Re}\langle g|A|f\rangle = 0$ and $\lambda = i \Rightarrow \mathscr{Im}\langle g|A|f\rangle = 0$

So $A = 0$ as required.

Theorem: I is hermitian

Proof: Preservation of the norm implies that $\forall|f\rangle \in \mathscr{F}$

18.2 $$\langle f|f\rangle = \langle f|\left(1 + i\frac{I^\dagger}{\chi}\right)\bar{\mu}\mu\left(1 - i\frac{I}{\chi}\right)|f\rangle$$

by 12.2. So

18.3 $$\langle f|f\rangle = |\mu|^2(\langle f|f\rangle + \langle f|I^\dagger I|f\rangle + i\langle f|I^\dagger - I|f\rangle)$$

Taking the imaginary part

18.4 $$\langle f|\frac{I^\dagger - I}{\chi}|f\rangle = 0$$

But $i(I^\dagger - I)$ is hermitian. So it follows from 18.1 that $I = I^\dagger$, so I is hermitian.

Proof: From 18.3 and 18.4, $\forall|f\rangle \in \mathscr{F}$

18.5 $$\frac{\langle f|I^2|f\rangle}{\langle f|f\rangle} = \frac{1 - |\mu|^2}{|\mu|^2}$$

But to maintain the interpretation that I describes discrete particle interactions we must regularise the interaction by prohibiting the possibility that two interactions take place at the same instant in the time line of a particle, or equivalently that a particle can be annihilated at the instant of its creation. Hence we have $I^2(x) = 0$ (regularisation introduces a non-linearity, but one which does not affect practical calculation). It follows immediately that $|\mu|^2 = 1$

Definition: A field is a mapping $\mathbb{R}^4 \otimes S \rightarrow \mathscr{F}$, where S is the set of spin indices introduced in section 13, *Continuous Wave Mechanics* and the elements of $\mathscr{F}$ are regarded as operators.

Definition: The field of creation operators for a particle in interaction is defined by

18.6 $$\forall(x, \alpha) = (x_0, \boldsymbol{x}, \alpha) \in \mathbb{R}^4 \otimes S. \quad |\underline{x, \alpha}\rangle : \mathscr{F} \rightarrow \mathscr{F}$$

The name field is used for historical reasons, not to suggest a physical field over classical space-time. We will find that photons are not created in an eigenstate of position (i.e. in a state labelled by a measurement of position), so we do not in general have $\forall x \in \mathrm{N}, |\underline{x, \alpha}\rangle = |x, \alpha\rangle$.

Definition: Let $|\underline{\alpha}\rangle = |\underline{0, \alpha}\rangle$ be the operator for the creation a particle at the origin.

Definition: By 10.8, the annihilation operator $\langle\underline{x, \alpha}| : \mathscr{F} \rightarrow \mathscr{F}$ is the hermitian conjugate.

Theorem: The creation operator $|\underline{x, \alpha}\rangle : \mathscr{F} \rightarrow \mathscr{F}$ for a particle at $(x, \alpha) = \mathbb{R}^4 \otimes S$ is given by

18.7 $$|\underline{x, \alpha}\rangle = \sum_r \eta(r) \int_{\mathrm{M}} d^3\boldsymbol{p} \langle \boldsymbol{p}, r|\underline{\alpha}\rangle e^{ip \cdot x} |\boldsymbol{p}, r\rangle$$

Proof: By the resolution of unity, 16.9, $|\underline{x,\alpha}\rangle:\mathscr{F}\to\mathscr{F}$ is given by

18.8 $$|\underline{x,\alpha}\rangle = \sum_r \eta(r)\int_M d^3\boldsymbol{p}\langle\boldsymbol{p},r|\underline{x,\alpha}\rangle|\boldsymbol{p},r\rangle$$

By the principle of homogeneity space-time translation maps the creation operators appearing in interactions into each other. Then, by 16.15,

18.9 $$\langle\boldsymbol{p},r|\underline{x,\alpha}\rangle = \langle\boldsymbol{p},r|\underline{\alpha}\rangle e^{ip\cdot x}$$

18.7 follows by substituting 18.9 into 18.8.

Definition: The derivative of the creation and annihilation operators is defined by differentiating 18.7.

18.10 $$\partial|\underline{x,\alpha}\rangle = |\partial\underline{x,\alpha}\rangle = \sum_{r=0}^{3}\eta(r)\int_M d^3\boldsymbol{p}\langle\boldsymbol{p},r|\underline{\alpha}\rangle ipe^{ip\cdot x}|\boldsymbol{p},r\rangle$$

There may be a number of different types of interaction, described by $I_j:\mathscr{F}\to\mathscr{F}$, where j runs over an index set. Let $e_j\in\mathbb{R}$ be the coupling constant for the interaction I_j. Only one type of interaction takes place at a time, but there is uncertainty about which. Under the identification of addition with quantum logical OR, the interaction operator $I(x_0):\mathscr{F}\to\mathscr{F}$, introduced in section 12, is

18.11 $$I = \sum_j e_j I_j$$

I is hermitian, and each I_j is independent by definition, so each I_j is hermitian.

Definition: In any finite discrete time interval, T, for each type of interaction, an operator,

18.12 $$H(x):\mathscr{F}\to\mathscr{F},$$

describes the interaction taking place at $x = (x_0,\boldsymbol{x})\in \mathrm{T}\otimes\mathrm{N}$, $H(x)$ is called interaction density.

The principle of homogeneity implies that $H(x)$ is the same, up to homomorphism, and has equal effect on a matter anywhere in N and for all times in T. I_j describes equal certainty that a particle interacts anywhere in N, so by the identification of addition with quantum logical OR I_j can be written as a sum

18.13 $$I_j(x_0) = \sum_{\boldsymbol{x}\in\mathrm{N}}\frac{1}{\chi^3}H(x_0,\boldsymbol{x}) = \sum_{\boldsymbol{x}\in\mathrm{N}}\frac{1}{\chi^3}H(x)$$

The sum in 18.13 is over space, but not necessarily over the spin index. Without loss of generality $H(x)$ is hermitian. By the definition of multiparticle space as a direct product (section 8), $H(x)$ can be factorised as a product of Hermitian operators, $J_\gamma(x)$, where γ runs over the particles in the interaction

18.14 $$H(x) = \prod_\gamma J_\gamma(x)$$

Definition: J is called a current operator. Its relationship to the electric current will be shown.

A number of particles participate in the interaction. As described by operators, the particles prior to interaction are annihilated and the particles present after interaction are created – a particle which is physically preserved is described as being annihilated and re-created. $H(x)$ can be represented as a Feynman node. Each line at the node corresponds to one particle in the interaction. In a single Feynman node there are no geometrical relationships with other matter, so it is not possible to say whether a particle's clock is running forwards or backwards with respect to the reference frame clock. So a line for the annihilation of a particle, γ, may also represent the creation of the corresponding antiparticle $\bar{\gamma}$.

Definition: Let $\langle \underline{x, \alpha}|$ be the annihilation operator for a particle at $(x, \alpha) = (x_0, \mathbf{x}, \alpha) \in \mathrm{T} \otimes \mathrm{N}$, and let $|\overline{x, \alpha}\rangle$ be the creation operator for the antiparticle. Then the particle field $\phi_\alpha(x) : \mathscr{F} \to \mathscr{F}$ is

18.15 $$\phi_\alpha(x =) |\overline{x, \alpha}\rangle + \langle \underline{x, \alpha}|$$

Then each line at the Feynman node corresponds to a particle field modelling the creation or annihilation of a particle. Clearly the hermitian conjugate of a particle field is the antiparticle field

18.16 $$\phi^\dagger{}_\alpha(x) = |\underline{x, \alpha}\rangle + \langle \overline{x, \alpha}|$$

In the general case $J_\gamma(x)$ is hermitian so it combines the particle and antiparticle fields

18.17 $$J_\gamma(x) = J_\gamma(\phi_\alpha(x), \phi^\dagger{}_\alpha(x))$$

Then the general form of the interaction is

18.18 $$I_j(x_0) = :\sum_{\mathbf{x} \in \mathrm{N}} \frac{1}{\chi^3} \prod_\gamma J_\gamma(|\overline{x, \alpha}\rangle + \langle \underline{x, \alpha}|, |\underline{x, \alpha}\rangle + \langle \overline{x, \alpha}|):$$

The colons reorder the creation and annihilation operators by placing all creation operators to the left of all annihilation operators, to ensure that false values are not generated corresponding to the annihilation of particles in the interaction in which they are created. Particular interactions can be postulated as operators with the general form of 18.18, we can examine whether the resulting theoretical properties correspond to the observed behaviour of matter.

Definition: Let π be the permutation such that $\tau_{\pi(n)} > \ldots \tau_{\pi(2)} > \tau_{\pi(1)}$ Then the time ordered product is

$$\mathrm{T}\{I(\tau_n)\ldots I(\tau_1)\} = I(\tau_{\pi(n)})\ldots I(\tau_{\pi(1)})$$

Theorem: (Locality)

18.19 $$\forall x, y \in \mathrm{T} \otimes \mathrm{N} \text{ such that } x - y \text{ is space-like } \langle [H(y), H(x)] \rangle = 0$$

Proof: Iterate 12.2 from an initial condition at $t = 0$ given by $|f\rangle_0 \in \mathscr{F}$

$$|f\rangle_1 = \mu\left(1 - i\frac{I(0)}{\chi}\right)|f\rangle_0$$

$$|f\rangle_2 = \mu^2\left(1 - i\frac{I(1)}{\chi}\right)\left(1 - i\frac{I(0)}{\chi}\right)|f\rangle_0$$

$$|f\rangle_3 = \mu^3\left(1 - i\frac{I(2)}{\chi}\right)\left(1 - i\frac{I(1)}{\chi}\right)\left(1 - i\frac{I(0)}{\chi}\right)|f\rangle_0$$

Expand after T iterations

18.20 $$|f\rangle_T = \mu^T\left(1 + \frac{i}{\chi}\sum_{\tau_1 = 0}^{T-1} I(\tau) + \frac{(-i)^2}{\chi^2}\sum_{\substack{\tau_2 = 0 \\ \tau_2 > \tau_1}}^{T-1} I(\tau_2)\sum_{\tau_1 = 0}^{T-1} I(\tau_1) + \ldots\right)|f\rangle_0$$

Then 18.20 is

18.21 $$|f\rangle_T = \mu^T\left(1 + \sum_{n=1}^{T}\frac{(-i)^n}{n!\chi^n}\sum_{\substack{\tau_1\ldots\tau_n = 0 \\ i \neq j \Rightarrow \tau_i \neq \tau_j}}^{T-1} \mathrm{T}\{I(\tau_n)\ldots I(\tau_1)\}\right)|f\rangle_0$$

There may be any number of particles in the initial state $|f\rangle_0 \in \mathcal{F}$ so 18.21 can be interpreted directly as a quantum logical statement meaning that, since an unknown number of interactions take place at unknown positions and unknown time, the final state is labelled as the weighted sum of the possibilities. This statement ceases to make sense in the limit $T \to \infty$, which forces $\mathrm{N}_S \to \mathbb{N}^3 \otimes S$ to ensure that particles remain in N. The expansion may reasonably be expected to diverge under such conditions, but there is no problem for finite values of T and bounded N. By 18.13, 18.21 is

18.22 $$|f\rangle_T = \mu^T \left(1 + \sum_{n=1}^{T} \frac{(-i)^n}{n!\chi^{4n}} \sum_{\substack{x^1 \dots x^n \in \mathrm{T} \otimes \mathrm{N}_S \\ i \neq j \Rightarrow x_0^i \neq x_0^j}} \mathrm{T}\{H(x^n)\dots H(x^1)\} \right) |f\rangle_0$$

Under Lorentz transformation of 18.22 the order of interactions, $H(x^i)$, can be changed in the time ordered product whenever $x^i - x^j$ is space-like. But this cannot affect the final state $|f\rangle_T$ for any $T \in \mathbb{N}$.

Corollary: By 18.14 H factorises and the locality condition applies to the current operators.

18.23 $$\forall x, y \in \mathrm{T} \otimes \mathrm{N}_S \text{ such that } x - y \text{ is space-like } \langle [J(y), J(x)] \rangle = 0$$

Corollary: The equal time commutator between an observable operator O such that $O(x) = O(H(x))$ and the interaction density H obeys the commutation relation

18.24 $$\forall \boldsymbol{x} \neq \boldsymbol{y}, [H(x), O(y)]_{x_0 = y_0} = 0$$

19 Classical Law

Theorem: In an inertial reference frame, momentum is conserved.

Proof: Classical momentum is the expectation of the momentum of a large number of particles, so it is sufficient to prove conservation of momentum in each particle interaction. The expectation of momentum is constant for each particle by Newton's first law, 16.4. Expand the interaction density, 18.18, as a sum of terms of the form

19.1 $$i(x_0) = \sum_{\boldsymbol{x} \in \mathrm{N}} h(x) = \sum_{\boldsymbol{x} \in \mathrm{N}} |\underline{x, \alpha}\rangle_1 \dots |\underline{x, \alpha}\rangle_m \langle \underline{x, \alpha}|_{m+1} \dots \langle \underline{x, \alpha}|_n$$

Where $|\underline{x, \alpha}\rangle_i$ and $\langle \underline{x, \alpha}|_i$ are creation and annihilation operators for the particles and antiparticles in the interaction, given by 18.7. Suppress the spin indices by writing $\forall \boldsymbol{p} \in \mathrm{M}\ \ s = 1, 2, 3, 4\ \ |\boldsymbol{p}\rangle = |\boldsymbol{p}, s\rangle$ and $|\underline{x}\rangle = |\underline{x, \alpha}\rangle$. We have from 19.1, $\forall n, m \in \mathbb{N}, n, m > 0$, $\forall$ plane wave $|\boldsymbol{p}^1\rangle, \dots, |\boldsymbol{p}^n\rangle$

$$\langle \boldsymbol{p}^1; \dots; \boldsymbol{p}^m | i(x_0) | \boldsymbol{p}^{m+1}; \dots; \boldsymbol{p}^n \rangle = \langle \boldsymbol{p}^1; \dots; \boldsymbol{p}^m | \sum_{\boldsymbol{x} \in \mathrm{N}} |\underline{x}\rangle^1 \dots |\underline{x}\rangle^m \langle \underline{x}|^{m+1} \dots \langle \underline{x}|^n | \boldsymbol{p}^{m+1}; \dots; \boldsymbol{p}^n \rangle$$

Then, by 9.23

$$\langle \boldsymbol{p}^1; \dots; \boldsymbol{p}^m | i(x_0) | \boldsymbol{p}^{m+1}; \dots; \boldsymbol{p}^n \rangle = \sum_{\boldsymbol{x} \in \mathrm{N}} \sum_{\pi} \varepsilon(\pi) \prod_{i=1}^{m} \langle \boldsymbol{p}^i | \underline{x} \rangle^{\pi(i)} \sum_{\pi'} \varepsilon(\pi') \prod_{j=m+1}^{n} \langle \underline{x} |^j \boldsymbol{p}^{\pi'(j)} \rangle$$

which is a sum of terms of the form

$$\sum_{\boldsymbol{x} \in \mathrm{N}} \prod_{i=1}^{m} \langle \boldsymbol{q}_i | \underline{x} \rangle_{\pi(i)} \prod_{j=m+1}^{n} \langle \underline{x} |_j \boldsymbol{p}_{\pi'(j)} \rangle .$$

Using 18.9 and permuting $\boldsymbol{p}_{\pi'(j)} \to \boldsymbol{p}_j$ this reduces to a sum of terms of the form

$$\sum_{x \in \mathbb{N}} \prod_{i=1}^{m} \langle \boldsymbol{q}^i | \underline{\alpha} \rangle e^{iq^i \cdot x} \prod_{j=1}^{n} \langle \underline{\alpha} | \boldsymbol{p}^j \rangle e^{-ip^j \cdot x} = \delta\left(\sum_{j=m+1}^{n} \boldsymbol{p}^j - \sum_{i=1}^{m} \boldsymbol{q}^i \right) \prod_{i=1}^{m} \langle \boldsymbol{q}^i | \underline{\alpha} \rangle e^{-iq_0^i x_0} \prod_{j=1}^{n} \langle \underline{\alpha} | \boldsymbol{p}_j \rangle e^{-p_0^j x_0}$$

by 7.12. Thus momentum is conserved in each particle interaction, and so is conserved universally by Newton's first law 16.4.

Remark: Conservation of momentum depends solely on the principle of homogeneity as expressed in 18.9, and the mathematical properties of multiparticle vector space imposed upon the labelling of states. Energy is not conserved in an individual interaction.

We are interested in changes in classical observable quantities. That is changes in the expectation, $\langle O \rangle$ of an observable, $O = O(x) = O(t, \boldsymbol{x})$, given by 11.10. According to $\mathbb{L}6$ all observable quantities are composed of interaction operators, which, by 18.18, can be decomposed into fields which are differentiable and covariant. Thus physically observable discrete values are obtained from covariant differentiable functions, and difference equations in the discrete quantities are obtained by integrating covariant differential equations over one chronon.

Theorem: The expectation of an observable operator $O(x) = O(H(x))$ obeys the differential equations

19.2
$$\partial_0 \langle O(x) \rangle = \frac{i}{\chi^3} \langle [H(x), O(x)] \rangle + \langle \partial_0 O(x) \rangle$$
$$\text{For } \alpha = 1, 2, 3 \qquad \partial_\alpha \langle O(x) \rangle = \langle \partial_\alpha O(x) \rangle$$

Proof: By 12.2

$$\langle O(t+1) \rangle = \langle f |_t \left(1 + i\frac{I(t+1)}{\chi} \right) O(t+1) |\mu|^2 \left(1 - i\frac{I(t+1)}{\chi} \right) | f \rangle_t$$
$$= \langle f |_t \frac{i}{\chi} [I(t+1), O(t+1)] + O(t+1) | f \rangle_t$$

by 11.13, and 18.2, since the state is an eigenstate of O and $|\mu|^2 = 1$. Then

$$\langle O(t+1) \rangle - \langle O(t) \rangle = \langle f |_{t+1} O(t+1) | f \rangle_{t+1} - \langle f |_t O(t) | f \rangle_t$$
$$= \langle f |_t \frac{i}{\chi} [I(t+1), O(t+1)] + O(t+1) | f \rangle_t - \langle f |_t O(t) | f \rangle_t$$

Then, using linearity of kets treated as operators and rearranging

19.3
$$\langle O(t+1) \rangle - \langle O(t) \rangle = \frac{i}{\chi} \langle [I(t+1), O(t+1)] \rangle + \langle O(t+1) - O(t) \rangle$$

Then the solution to 19.3 is the restriction to integer values of the solution of

19.4
$$\partial_0 \langle O(x) \rangle = i \langle [I(t), O(x)] \rangle + \langle \partial_0 O(x) \rangle$$

Using locality, 18.24, with $x_0 = y_0$ 19.4 is

19.5
$$\partial_0 \langle O(x) \rangle = i \left\langle \left[\sum_{\boldsymbol{y} \in \mathbb{N}} \frac{1}{\chi^3} H(x_0, \boldsymbol{y}), O(x) \right] \right\rangle + \langle \partial_0 O(x) \rangle$$

Using locality, 18.19, 19.5 reduces to the time-component of 18.24. The proof of the space-like components is identical, but the commutator is zero because space translation is a homomorphism.

Corollary: Particles are point-like.

Note: Position is only a numerical value derived from a configuration of matter in measurement, and it is not obvious that this requires that particles are themselves point-like.

Proof: By 19.2 changes in $\langle O(x)\rangle$ have no dependence on particle interactions except at the point x.
Corollary: No observable particle effect may propagate faster than the speed of light.
Proof: By 19.2 $O(x)$ has no space-like dependence on particle interactions for any space-like slice.

18.24 involves the commutation relation between the interaction density, H, and the observable, O. By $\mathbb{L}6$ any observable operator is a combination of interaction operators, so observables are a combination of particle fields. Then 18.24 requires the commutators for particle fields. For fermions the creation operators anticommute, but commutation relations are obtained if the current, 18.17, is a composition of an even number of fermion fields.

20 The Photon Field

Photons are bosons, and having zero mass, the photon is its own antiparticle and $|\overline{x,\alpha}\rangle = |\underline{x,\alpha}\rangle$.
Definition: By 18.15, the photon field is

20.1 $$A_\alpha(x) = |\underline{x,\alpha}\rangle + \langle\underline{x,\alpha}|$$

which is hermitian, so only one photon field is necessary in the current, so $J = A$ is permissible and photons can be absorbed and emitted singly. The commutator is

20.2 $$[A_\alpha(x), A_\beta(y)] = [|\underline{x,\alpha}\rangle + \langle\underline{x,\alpha}|, |\underline{y,\beta}\rangle + \langle\underline{y,\beta}|] = \langle\underline{x,\alpha}|\underline{y,\beta}\rangle - \langle\underline{y,\beta}|\underline{x,\alpha}\rangle$$

Thus, by 16.10 and 18.9

20.3 $$[A_\alpha(x), A_\beta(y)] = \sum_r \eta(r)\int_M d^3\boldsymbol{p}\langle\underline{\alpha}|\boldsymbol{p}, r\rangle e^{-ip\cdot(x-y)}\langle\boldsymbol{p}, r|\underline{\beta}\rangle - \langle\underline{\beta}|\boldsymbol{p}, r\rangle e^{ip\cdot(x-y)}\langle\boldsymbol{p}, r|\underline{\alpha}\rangle$$

By $\mathbb{L}6$ the constraint that $A_\alpha(x)$ contains only components of spin α is necessary if the interaction operator creates eigenstates of spin. This is observed; we assume that it also holds for time-like and longitudinal spin. Then $\langle\underline{\alpha}|\boldsymbol{p}, r\rangle$ transforms as $w_\alpha(\boldsymbol{p}, r)$ (defined in 15.1) under space inversion. So

20.4 $$\langle\underline{\beta}|-\boldsymbol{p}, r\rangle\langle-\boldsymbol{p}, r|\underline{\alpha}\rangle = \langle\underline{\alpha}|\boldsymbol{p}, r\rangle\langle\boldsymbol{p}, r|\underline{\beta}\rangle$$

since $w_\alpha(\boldsymbol{p}, 0)$ has no space-like component and for $r = 1,2,3$ $w_\alpha(\boldsymbol{p}, r)$ has no time like component. Now substitute $\boldsymbol{p} \to -\boldsymbol{p}$ in the second term of 20.3 at $x_0 = y_0$

20.5 $$[A(x), A(y)]_{x_0 = y_0} = 0$$

Then by substituting $O = A$ in 18.24, and noting that, by 18.14, the commutation relationship with the interaction density is determined by the commutation relationship with the current

20.6 $$\partial_\alpha\langle A_\beta(x)\rangle = \langle\partial_\alpha A_\beta(x)\rangle$$

The physical interpretation of 20.6 is that since photons can be absorbed or emitted singly the number of photons cannot be an eigenstate of an operator constructed from the interaction and cannot therefore be known. So observable effects associated with photons depend only on changes in photon number. Let $\phi_\mu(x)$ be a a gauge term, that is an arbitrary solution of $\partial_\mu\phi_\mu(x) = 0$ having no physical meaning. Then physical predictions from 20.6 are invariant under the gauge transformation $A(x) \to A(x) + \phi(x)$, and the value of $\langle A(x)\rangle$ is hidden by the gauge term. Differentiating 20.6 using 18.24 gives

20.7 $$\partial^2\langle A(x)\rangle = \partial_\alpha\langle\partial_\alpha A(x)\rangle = i\langle[H(x), \partial_0 A(x)]\rangle + \langle\partial^2 A(x)\rangle$$

Differentiate twice and observe that $p^2 = 0$ for the photon so $\partial^2|\underline{x,\alpha}\rangle = 0$. Then from 20.1

20.8 $$\partial^2 A(x) = 0$$

Then 20.7 reduces to

20.9 $$\partial^2\langle A(x)\rangle = i\langle[H(x), \partial_0 A(x)]\rangle$$

Given H, 20.9 can be calculated from the commutator between the fields

20.10 $$[\partial A_\alpha(x), A_\beta(y)] = \langle\partial\underline{x, \alpha}|\underline{y, \beta}\rangle - \langle\underline{y, \beta}|\partial\underline{x, \alpha}\rangle$$

But by 18.10 and 18.9

20.11 $$\langle\partial\underline{x, \alpha}|\underline{y, \beta}\rangle = -\sum_{r=0}^{3}\eta(r)\int_M d^3\boldsymbol{p}\langle\underline{\alpha}|\boldsymbol{p}, r\rangle\langle\boldsymbol{p}, r|\underline{\beta}\rangle ipe^{-ip\cdot(x-y)}$$

and

20.12 $$\langle\underline{y, \beta}|\partial\underline{x, \alpha}\rangle = \sum_{r=0}^{3}\eta(r)\int_M d^3\boldsymbol{p}\langle\underline{\beta}|\boldsymbol{p}, r\rangle\langle\boldsymbol{p}, r|\underline{\alpha}\rangle ipe^{ip\cdot(x-y)}$$

Substituting $\boldsymbol{p} \to -\boldsymbol{p}$ in 20.12 at $x_0 = y_0$ and using 20.4 and 20.10 gives, for the space-like components of the derivative

20.13 $$\text{For } i = 1, 2, 3,\ [\partial_i A(x), A(y)]_{x_0 = y_0} = 0$$

and for the time-like component

20.14 $$[\partial_0 A_\alpha(x), A_\beta(y)]_{x_0 = y_0} = -2i\sum_{r=0}^{3}\eta(r)\int_M d^3\boldsymbol{p}\langle\underline{\alpha}|\boldsymbol{p}, r\rangle\langle\boldsymbol{p}, r|\underline{\beta}\rangle p_0 e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}$$

Theorem: The commutator 20.14 is Lorentz covariant and satisfies locality, 18.23, if

20.15 $$\langle\underline{\alpha}|\boldsymbol{p}, r\rangle = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}}\frac{w_\alpha(\boldsymbol{p}, r)}{\sqrt{2p_0}}$$

Proof: It follows from 20.15 that

20.16 $$\sum_{r=0}^{3}\eta(r)\langle\underline{\alpha}|\boldsymbol{p}, r\rangle\langle\boldsymbol{p}, r|\underline{\beta}\rangle = \frac{g_{\alpha\beta}}{16\pi^3 p_0}$$

where g is given by 13.18. Then substituting 20.16 into 20.14, and using 7.5 establishes that locality is satisfied by the equal time commutation relation

20.17 $$[\partial_0 A(x), A(y)]_{x_0 = y_0} = -ig\delta_{\boldsymbol{xy}}$$

Substituting 20.15 into 20.1 using 16.3 gives the photon field

20.18 $$A_\alpha(x) = \sum_{r=0}^{3}\eta(r)\int_M\frac{d^3\boldsymbol{p}}{\sqrt{2p_0}}(e^{ip\cdot x}|\boldsymbol{p}, r\rangle + e^{-ip\cdot x}\langle\boldsymbol{p}, r|)w_\alpha(\boldsymbol{p}, r)$$

By 20.16, 18.7 and 16.11

20.19 $$\langle\underline{x, \alpha}|\underline{y, \beta}\rangle = \frac{\chi^3 g_{\alpha\beta}}{8\pi^3}\int_M\frac{d^3\boldsymbol{p}}{2p_0}e^{-ip\cdot(x-y)}$$

So the commutator, 20.2, is

20.20 $$[A_\alpha(x), A_\beta(y)] = \frac{\chi^3 g_{\alpha\beta}}{8\pi^3}\int_M\frac{d^3\boldsymbol{p}}{2p_0}(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)})$$

It is a text book result, e.g. [25], that 20.20 is Lorentz covariant and zero outside the light cone.

Theorem: $\langle A(x)\rangle$ satisfies the Lorentz gauge condition

20.21 $$\partial_\alpha \langle A_\alpha(x)\rangle = 0$$

Proof: by 20.6

$$\partial_\alpha \langle A_\alpha(x)\rangle = \langle \partial_\alpha A_\alpha(x)\rangle$$

$$= \langle \sum_{r=0}^{3} \eta(r) \int_M \frac{d^3\boldsymbol{p}}{\sqrt{2p_0}} (e^{ip\cdot x}|\boldsymbol{p}, r\rangle + e^{-ip\cdot x}\langle \boldsymbol{p}, r|) i(p_\alpha - p_\alpha) w_\alpha(\boldsymbol{p}, r)\rangle$$

by differentiating 20.18. But this is zero which establishes 20.21.

21 The Dirac Field

Definition: By 18.15, the Dirac field is

21.1 $$\psi_\alpha(x) = |\overline{x, \alpha}\rangle + \langle \underline{x, \alpha}|$$

We know from observation that a Dirac particle can be an eigenstate of position. So, by $\mathbb{L}6$, it is possible to form the position operator 11.1 from the current 18.17, for any region X which can be as small as the apparatus will allow. Position kets are a basis, so 11.1 reduces to 11.9 up to the resolution of the apparatus. Current can only generate eigenstates of spin and position if it does not mix basis states, so

21.2 $$\forall x \in N \ \ |\underline{x, \alpha}\rangle = |x, \alpha\rangle$$

Then by 16.1

21.3 $$\langle \underline{\alpha}| \boldsymbol{p}, r\rangle = \left(\frac{\chi}{2\pi}\right)^{\frac{3}{2}} u_\alpha(\boldsymbol{p}, r)$$

and by 18.8

21.4 $$\langle \underline{x, \alpha}| = \left(\frac{\chi}{2\pi}\right)^{\frac{3}{2}} \sum_r \int_M d^3\boldsymbol{p}\, u_\alpha(\boldsymbol{p}, r) e^{-ip\cdot x}\langle \boldsymbol{p}, r|$$

Definition: The Dirac adjoint of the annihilation operator $\langle \underline{x, \alpha}|$ is

21.5 $$|\underline{x, \hat{\alpha}}\rangle = \sum_\mu |\underline{x, \mu}\rangle \gamma^0_{\mu\alpha} = \left(\frac{\chi}{2\pi}\right)^{\frac{3}{2}} \sum_r \int_M d^3\boldsymbol{p}\, \hat{u}_\alpha(\boldsymbol{p}, r) e^{ip\cdot x}|\boldsymbol{p}, r\rangle$$

Similarly by 16.2

21.6 $$\langle \overline{\alpha}| \boldsymbol{p}, r\rangle = \left(\frac{\chi}{2\pi}\right)^{\frac{3}{2}} \bar{v}_\alpha(\boldsymbol{p}, r)$$

and by 18.8

21.7 $$|\overline{x, \alpha}\rangle = \left(\frac{\chi}{2\pi}\right)^{\frac{3}{2}} \sum_r \int_M d^3\boldsymbol{p}\, v_\alpha(\boldsymbol{p}, r) e^{ip\cdot x}|\boldsymbol{p}, r\rangle$$

Definition: The Dirac adjoint of the creation operator $|\overline{x, \alpha}\rangle$ is

21.8 $$\langle \overline{x, \hat{\alpha}}| = \sum_\mu \langle \overline{x, \mu}|\gamma^0_{\mu\alpha} = \left(\frac{\chi}{2\pi}\right)^{\frac{3}{2}} \sum_r \int_M d^3\boldsymbol{p}\, \hat{v}_\alpha(\boldsymbol{p}, r) e^{ip\cdot x}|\boldsymbol{p}, r\rangle$$

Definition: The Dirac adjoint of the field is

21.9 $$\hat{\psi}_\alpha(x) = \psi^\dagger{}_\mu(x)\gamma^0_{\mu\alpha} = |\underline{x, \hat{\alpha}}\rangle + \langle \overline{x, \hat{\alpha}}|$$

Theorem: The anticommutation relations for the Dirac field and Dirac adjoint and obey

21.10 $$\{\psi_\nu(x), \psi_\lambda(y)\} = \{\hat{\psi}_\mu(x), \hat{\psi}_\kappa(y)\} = 0$$

21.11 $$\{\psi_\alpha(x), \hat{\psi}_\beta(y)\}_{x_0 = y_0} = \chi^3 \gamma^0_{\alpha\beta} \delta_{\boldsymbol{xy}}$$

Proof: 21.10 follows from the definitions, 21.1 and 21.9 By 10.10 and 9.17 we have

21.12 $$\{\psi_\alpha(x), \hat{\psi}_\beta(y)\} = \{\langle \underline{x, \alpha}|, |\underline{y, \hat{\beta}}\rangle\} + \{|\overline{x, \alpha}\rangle, \langle \overline{y, \hat{\beta}}|\} = \langle \underline{x, \alpha}|\underline{y, \hat{\beta}}\rangle + \langle \overline{y, \hat{\beta}}|\overline{x, \alpha}\rangle^{\mathrm{T}}$$

where $^{\mathrm{T}}$ denotes that α and β are transposed.
By 21.4 and 21.5, and using 16.11.

$$\langle \underline{x, \alpha}|\underline{y, \hat{\beta}}\rangle = \frac{\chi^3}{8\pi^3} \sum_r \int_{\mathrm{M}} d^3\boldsymbol{p}\, u_\alpha(\boldsymbol{p}, r) \hat{u}_\beta(\boldsymbol{p}, r) e^{-ip \cdot (x-y)}$$

21.13 $$= \frac{\chi^3}{8\pi^3} \int_{\mathrm{M}} \frac{d^3\boldsymbol{p}}{2p_0} (p \cdot \gamma + m)_{\alpha\beta} e^{-ip \cdot (x-y)}$$

by 14.11. Likewise for the antiparticle, by 21.7 and 21.8

$$\langle \overline{y, \hat{\beta}}|\overline{x, \alpha}\rangle^{\mathrm{T}} = \frac{\chi^3}{8\pi^3} \sum_r \int_{\mathrm{M}} d^3\boldsymbol{p}\; v_\alpha(\boldsymbol{p}, r) \hat{v}_\beta(\boldsymbol{p}, r)\; e^{ip \cdot y - ix \cdot p}$$

21.14 $$= \frac{\chi^3}{8\pi^3} \int_{\mathrm{M}} \frac{d^3\boldsymbol{p}}{2p_0}\; (p \cdot \gamma - m)_{\alpha\beta}\; e^{ip \cdot (x-y)}$$

by 14.19. Substituting $\boldsymbol{p} \to -\boldsymbol{p}$ at $x_0 = y_0$ in 21.14 gives

21.15 $$\langle \overline{y, \hat{\beta}}|\overline{x, \alpha}\rangle_{x_0 = y_0} = \frac{\chi^3}{8\pi^3} \int_{\mathrm{M}} \frac{d^3\boldsymbol{p}}{2p_0}\; (2p_0\gamma^0 - p \cdot \gamma - m) e^{-i\boldsymbol{p} \cdot (\boldsymbol{x} - \boldsymbol{y})}$$

So, by 21.12, adding 21.13 and 21.15 at $x_0 = y_0$ gives the equal time anticommutator

21.16 $$\{\psi_\alpha(x), \hat{\psi}_\beta(y)\}_{x_0 = y_0} = \frac{\chi^3}{8\pi^3} \gamma^0_{\alpha\beta} \int_{\mathrm{M}} d^3\boldsymbol{p}\;\; e^{-i\boldsymbol{p} \cdot (\boldsymbol{x} - \boldsymbol{y})}$$

21.11 follows from 7.5.

Theorem: The anticommutation relations for the Dirac field and the Dirac adjoint obey locality, 18.23, and are Lorentz covariant.

Proof: By 21.13

21.17 $$\langle \underline{x, \alpha}|\underline{y, \hat{\beta}}\rangle = \frac{\chi^3}{8\pi^3} (i\partial \cdot \gamma + m) \int_{\mathrm{M}} \frac{d^3\boldsymbol{p}}{2p_0} e^{-ip \cdot (x-y)}$$

And by 21.14

21.18 $$\langle \overline{y, \hat{\beta}}|\overline{x, \alpha}\rangle^{\mathrm{T}} = -\frac{\chi^3}{8\pi^3} (i\partial \cdot \gamma + m) \int_{\mathrm{M}} \frac{d^3\boldsymbol{p}}{2p_0}\;\; e^{ip \cdot (x-y)}$$

By 21.12 the anticommutator is found by adding 21.17 and 21.18

21.19 $$\{\psi_\alpha(x), \hat{\psi}_\beta(y)\} = \frac{\chi^3}{8\pi^3} (i\partial \cdot \gamma + m) \int_{\mathrm{M}} \frac{d^3\boldsymbol{p}}{2p_0} (e^{-ip \cdot (x-y)} - e^{ip \cdot (x-y)})$$

It is a text book result, e.g. [25], that 21.19 is Lorentz covariant and zero outside the light cone.

22 The Electromagnetic Interaction

Under $\mathbb{L}7$ we postulate the intuitively appealing minimal interaction characterised by the emission or absorption of a photon by a Dirac particle. According to 18.14 an interaction H between photons and Dirac particles is described by a combination of particle currents, which, by 18.17, are themselves hermitian combinations of particle fields.

Definition: The photon current operator is $A(x)$

Definition: The Dirac current operator is

22.1 $$j_\alpha(x) \;=\; :\hat{\psi}_\mu(x)\gamma^\alpha_{\mu\nu}\psi_\nu(x): \;=\; :\hat{\psi}(x)\gamma^\alpha\psi(x):$$

Lemma: The Dirac current is hermitian

Proof: By the definitions 22.1 and 21.9, and using 14.8.

22.2 $$j^\dagger(x) \;=\; :\psi^\dagger(x)\gamma^{\alpha\dagger}\gamma^0\psi(x): \;=\; :\psi^\dagger(x)\gamma^0\gamma^\alpha\psi(x): \;=\; :\hat{\psi}(x)\gamma^\alpha\psi(x): \;=\; j(x)$$

Postulate (under $\mathbb{L}7$): Let e be the electromagnetic coupling constant. The electromagnetic interaction density is

22.3 $$H(x) \;=\; ej(x)\cdot A(x) \;=\; e:\hat{\psi}(x)\gamma\cdot A(x)\psi(x):$$

Lemma:

22.4 $$\langle\partial\;\cdot j(x)\rangle \;=\; 0$$

Proof: Using the definitions 21.1 and 21.9 to expand 22.1

22.5 $$j_\alpha(x) \;=\; |\underline{x\,,\hat{\mu}}\rangle\gamma^\alpha_{\mu\nu}|\overline{x,\nu}\rangle + |\underline{x\,,\hat{\mu}}\rangle\gamma^\alpha_{\mu\nu}\langle\underline{x,\nu}| - \gamma^\alpha_{\mu\nu}|\overline{x,\nu}\rangle\langle\overline{x\,,\hat{\mu}}| + \langle\overline{x\,,\hat{\mu}}|\gamma^\alpha_{\mu\nu}\langle\underline{x,\nu}|$$

where the summation convention is used for the repeated indices, μ and ν. In classical situations we only consider states of a definite number of Dirac particles, so the expectation of the pair creation and annihilation terms is zero by 8.2. Using 21.4 and 21.5 and differentiating the particle term in 22.5

$$\partial_\alpha|\underline{x\,,\hat{\mu}}\rangle\gamma^\alpha_{\mu\nu}\langle\underline{x,\nu}| \;=\; \frac{\chi^3}{8\pi^3}\sum_{r,s}\int_M d^3\boldsymbol{p}\int_M d^3\boldsymbol{q}\,i\hat{u}(\boldsymbol{p},r)(q\cdot\gamma-p\cdot\gamma)u(\boldsymbol{q},s)e^{ix\cdot(q-p)}|\boldsymbol{p},r\rangle\langle\boldsymbol{q},s|$$

Using 21.7 and 21.8 and differentiating the antiparticle term in 22.5

$$\partial_\alpha\gamma^\alpha_{\mu\nu}|\overline{x,\nu}\rangle\langle\overline{x\,,\hat{\mu}}| \;=\; \frac{\chi^3}{8\pi^3}\sum_{r,s}\int_M d^3\boldsymbol{p}\int_M d^3\boldsymbol{q}\,i\hat{v}(\boldsymbol{q},r)(p\cdot\gamma-q\cdot\gamma)v(\boldsymbol{p},s)e^{ix\cdot(p-q)}|\boldsymbol{p},r\rangle\langle\boldsymbol{q},s|$$

Here v and $\hat{v}$ have been ordered so that the spin index can be unambiguously omitted. 22.4 follows by differentiating 22.5 and using 14.9 and 14.18.

Lemma:

22.6 $$[j_0(x), j_\alpha(x)] \;=\; 0$$

Proof:
$$[\psi(x), j_\alpha(x)] \;=\; [\psi(x), :\hat{\psi}(x)\gamma^\alpha\psi(x):]$$
$$=\; \{\psi(x),\hat{\psi}(x)\}\gamma^\alpha\psi(x)$$

22.7 $$=\; \chi^3\gamma^0\gamma^\alpha\psi(x)$$ by 21.11.

Take the hermitian conjugate and apply 14.8

$$[j_\alpha(x),\psi^\dagger(x)] \;=\; \chi^3\psi^\dagger(x)\gamma^{\alpha\dagger}\gamma^0 \;=\; \chi^3\hat{\psi}(x)\gamma^\alpha$$

Postmultiply by γ^0

22.8 $$[j_\alpha(x),\hat{\psi}(x)] \;=\; \chi^3\hat{\psi}_\mu(x)\gamma^\alpha\gamma^0$$

So, by commuting the terms,

$$
\begin{aligned}
[j_0(x), j_\alpha(x)] &= [:\hat{\psi}(x)\gamma^0\psi(x):, j_\alpha(x)] \\
&= \hat{\psi}(x)\gamma^0[\psi(x), j_\alpha(x)] + [\hat{\psi}(x), j_\alpha(x)]\gamma^0\psi(x) \\
&= \chi^3\hat{\psi}(x)\gamma^0\gamma^0\gamma^\alpha\psi(x) - \chi^3\hat{\psi}(x)\gamma^\alpha\gamma^0\gamma^0\psi(x)
\end{aligned}
$$

using 22.7 and 22.8. 22.6 follows from 14.8

Theorem: $\langle j \rangle$ is a classical conserved current, i.e.

22.9 $$\partial \cdot \langle j(x) \rangle = 0$$

Proof: Substituting $O = j_\alpha$ in 19.2

22.10 $$\partial_a \langle j_\alpha(x) \rangle = = i\langle [H(x), j_0(x)] \rangle + \langle \partial_\alpha j_\alpha(x) \rangle$$

22.9 follows from 22.4 and 22.6, so $\langle j \rangle$ is conserved.

Theorem: $\langle j_0 \rangle$ can be identified with classical electric charge density, i.e.

22.11 $$\forall | f \rangle \in \mathscr{F},\ \langle j_0(x) \rangle = |\langle x | f \rangle|^2 - |\langle f | \overline{x} \rangle|^2$$

Proof: It is straightforward from 10.2 that j is additive for multiparticle states, so it is sufficient to show the theorem for a one particle state $| f \rangle \in \mathbb{H}$. By 22.5

$$
\begin{aligned}
\langle j_0(x) \rangle &= \langle f | \underline{x, \hat{\mu}} \rangle \gamma^0_{\mu\nu} \langle \underline{x, \nu} | f \rangle - \gamma^0_{\mu\nu} \langle f | \overline{x, \nu} \rangle \langle \overline{x, \hat{\mu}} | f \rangle \\
&= \langle f | \underline{x} \rangle \gamma^0\gamma^0 \langle \underline{x} | f \rangle - \langle \overline{x} | f \rangle \gamma^0\gamma^0 \langle f | \overline{x} \rangle
\end{aligned}
$$

by ordering terms so that the spinor indices can be suppressed. Then 21.13 follows from 21.2 and 14.8

Except in so far as $\mathbb{A}2$ was used to justify an analysis of measurement, classical law does not form part of the assumptions, and according to $\mathbb{L}7$, the claim that the minimal interaction is the cause of the electromagnetic force requires:

Theorem: $\langle A(x) \rangle$ satisfies Maxwell's Equations

22.12 $$\partial^2 \langle A_\alpha(x) \rangle - \partial_\alpha \partial_\mu \langle A_\mu(x) \rangle = -e\langle j_\alpha(x) \rangle$$

Corollary: Maxwell's equations simplify immediately to their form in the Lorentz gauge

22.13 $$\partial^2 \langle A(x) \rangle = -e\langle j(x) \rangle$$

Proof: By 20.21 it is sufficient to prove the corollary. By 20.9 and 22.3

22.14 $$\partial^2 \langle A(x) \rangle = i\langle [j(x) \cdot A(x), \partial_0 A(x)] \rangle$$

22.13 follows immediately from 20.17.

Theorem: (Classical gauge invariance). Let g be an arbitrary differentiable function. Then observable results are invariant under gauge transformation of the photon field given by

22.15 $$\langle A_\alpha(x) \rangle \to \langle A_\alpha(x) + \partial_\alpha g(x) \rangle = \langle A_\alpha(x) \rangle + \partial_\alpha g(x)$$

Proof: It is a well known result following from 22.12 that the classical properties of the electromagnetic field depend only on derivatives of $\langle A(x) \rangle$, defined by

22.16 $$F_{\alpha\beta} \equiv \partial_\alpha \langle A_\beta(x) \rangle - \partial_\beta \langle A_\alpha(x) \rangle$$

Then $F_{\alpha\beta}$ is clearly invariant under 22.15. Although classical electrodynamics is gauge invariant, the Lorentz gauge, 20.21, is theoretically determined and we have $\partial_\alpha g = 0$.

23 Feynman Rules

12.2 does not allow interactions to be "switched off", but for a stable particle in isolation the ket is not changed by interactions. This cannot be true of a single particle ket in the sense of section 8, *Multiparticle States*, but it can be true if the state includes both the particle itself and a cloud of 'virtual particles' such that the effect of the creation and annihilation of particles in the interaction is to leave the ket unchanged. Any ambiguity between kets for physical states (including virtual particles), and the definition of kets given in section 5, *Ket Space*, is resolved because the virtual particles are not measured. Thus $\exists m \in \mathbb{R}$ such that the law of evolution, 12.2, of the ket for a particle in isolation is

23.1 $$|f\rangle_{t+1} = \mu\left(1 - i\frac{I(t)}{\chi}\right)|f\rangle_t = e^{im}|f\rangle_t$$

Definition: m is the rest mass of a particle.

Comparison of 23.1 with 12.4 shows that the inclusion of virtual particles renormalises the theory to use rest mass instead of bare mass. We can apply the renormalised theory to scattering experiments, where the initial and final states consist of particles in isolation, including virtual particles, and where the time taken for a stable cloud of virtual particles to develop from the creation of a single particle may be ignored.

Definition: For any vector p, such that $p^2 = m^2$, let $\tilde{p} = (\tilde{p}_0, \boldsymbol{p})$ be a matrix for any $\tilde{p}_0 \in \mathbb{R}$. $\tilde{p}$ satisfies the identity

23.2 $$\tilde{p}_0^2 - p_0^2 \equiv \tilde{p}^2 - m^2$$

Lemma: For $x > 0$, $\varepsilon > 0$ we have the identities

23.3 $$\frac{e^{i(p_0 - i\varepsilon)x}}{2(p_0 - i\varepsilon)} \equiv \frac{-i}{2\pi}\int_{-\infty}^{\infty} d\tilde{p}_0 \frac{e^{-i\tilde{p}_0 x}}{\tilde{p}_0^2 - (p_0 - i\varepsilon)^2} \equiv \frac{-i}{2\pi}\int_{-\infty}^{\infty} d\tilde{p}_0 \frac{e^{-i\tilde{p}_0 x}}{\tilde{p}^2 - m^2 + 2ip_0\varepsilon + \varepsilon^2}$$

23.4 $$\frac{e^{i(p_0 - i\varepsilon)x}}{2} \equiv \frac{-i}{2\pi}\int_{-\infty}^{\infty} d\tilde{p}_0 \frac{\tilde{p}_0 e^{-i\tilde{p}_0 x}}{\tilde{p}^2 - m^2 + 2ip_0\varepsilon + \varepsilon^2}$$

and for $x < 0$ $\varepsilon > 0$ we have the identities

23.5 $$\frac{e^{-i(p_0 - i\varepsilon)x}}{2(p_0 - i\varepsilon)} \equiv \frac{-i}{2\pi}\int_{-\infty}^{\infty} d\tilde{p}_0 \frac{e^{-i\tilde{p}_0 x}}{\tilde{p}^2 - m^2 + 2ip_0\varepsilon + \varepsilon^2}$$

23.6 $$\frac{e^{-i(p_0 - i\varepsilon)x}}{2} \equiv \frac{-i}{2\pi}\int_{-\infty}^{\infty} d\tilde{p}_0 \frac{\tilde{p}_0 e^{-i\tilde{p}_0 x}}{\tilde{p}^2 - m^2 + 2ip_0\varepsilon + \varepsilon^2}$$

Proof: These are straightforward contour integrals and the proofs are left to the reader.

Definition: The step function is given $\forall x \in \mathbb{R}$, by

23.7 $$\Theta(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 & \text{if } x > 0 \end{cases}$$

Let $|g\rangle \in \mathscr{F}$ be a measured state at time T. $\langle g|f\rangle_T$ can be evaluated iteratively from 18.22 by using 10.5. The result is the sum of the terms generated by the braket between $\langle \underline{x^n, \alpha}|$ and every earlier creation operator $|\underline{x^j, \alpha}\rangle$ and every particle in $|f\rangle_0$, and the braket between $|\underline{x^n, \alpha}\rangle$ and every later annihilation operator $\langle \underline{x^j, \alpha}|$ and every particle in the final state $\langle g|$ (all other brakets are zero). This procedure is repeated for every creation and annihilation operator in 22.5, and for every term in 18.22. To keep check on the brakets so formed, each factor $I_j(x_0)$ in 18.18 is represented as a Feynman node.

Each line at the node corresponds to one of the particles in the interaction and to one of the particle fields $|\overline{x, \alpha}\rangle + \langle \underline{x, \alpha}|$ in $I_j(x_0)$. Then when the braket is formed the corresponding connection between the nodes is made in a diagram. Each internal connecting line, or propagator, is associated with a particular particle type. Photons are denoted by wavy lines, and Dirac particles by arrowed lines, so that for particles the arrow is in the direction of time ordering in 18.22, and for antiparticles the arrow is opposed to the time ordering. In this way all time ordered diagrams are formed by making each possible connection, from the creation of a particle to the annihilation of a particle of the same type, and we calculate rules to evaluate the diagram from 18.22. There is an overall factor $1/n!$ for a diagram with n vertices. The vertices, x^n, are such that $n \neq j \Rightarrow x_0^n \neq x_0^j$ and, by examination of 18.22 and 22.3, generate the expression

23.8 $$\mu e \sum_{x^n \in \mathrm{T} \otimes \mathrm{N}} \frac{-i\gamma}{\chi^4}$$

The initial and final states must be expressed as plane wave expansions so that the time invariant inner product 16.10 can be used. But plane waves span $\mathscr{T}$, so without loss of generality we can use plane wave states for the initial and final states. Then each initial particle in the state $|\boldsymbol{p}, r\rangle$ connected to the node x^n gives, from 18.9

$$\langle \underline{x^n, \alpha}|\boldsymbol{p}, r\rangle = \langle \underline{\alpha}|\boldsymbol{p}, r\rangle e^{-ip \cdot x^n}$$

So

23.9 $$\langle \underline{x^n, \alpha}|\boldsymbol{p}, r\rangle = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \frac{w_\alpha(\boldsymbol{p}, r)}{\sqrt{2p_0}} e^{-ip \cdot x^n}$$ for a photon, by 20.15

23.10 $$\langle \underline{x^n, \alpha}|\boldsymbol{p}, r\rangle = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} u_\alpha(\boldsymbol{p}, r) e^{-ip \cdot x^n}$$ for a Dirac particle, by 21.3

23.11 $$\langle \overline{x^n, \hat{\alpha}}|\boldsymbol{p}, r\rangle = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \hat{v}_\alpha(\boldsymbol{p}, r) e^{-ip \cdot x^n}$$ for an antiparticle, by 21.6

Similarly for each final particle in the state $\langle \boldsymbol{p}, r|$ connected to the node x^n we have

23.12 $$\left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \frac{w_\alpha(\boldsymbol{p}, r)}{\sqrt{2p_0}} e^{ip \cdot x^n}$$ for a photon, by 20.15

23.13 $$\left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \hat{u}_\alpha(\boldsymbol{p}, r) e^{ip \cdot x^n}$$ for a Dirac particle, by 21.3

23.14 $$\left(\frac{1}{2\pi}\right)^{\frac{3}{2}} v_\alpha(\boldsymbol{p}, r) e^{ip \cdot x^n}$$ for an antiparticle, by 21.6

Then we have an expression for the photon propagator

23.15 $$\Theta(x_0^n - x_0^j)\langle \underline{x^n, \alpha}|\underline{x^j, \beta}\rangle + \Theta(x_0^j - x_0^n)\langle \underline{x^j, \beta}|\underline{x^n, \alpha}\rangle^{\mathrm{T}}$$

By 20.19 this is

23.16 $$\frac{\chi^3 g_{\alpha\beta}}{8\pi^3} \int_{\mathrm{M}} \frac{d^3\boldsymbol{p}}{2p_0} [\Theta(x_0^n - x_0^j) e^{-ip \cdot (x^n - x^j)} + \Theta(x_0^j - x_0^n) e^{ip \cdot (x^n - x^j)}]$$

Use 23.3 in the first term, recalling that $m^2 = 0$, and use 23.5 and substitute $\boldsymbol{p} \rightarrow -\boldsymbol{p}$ in the second term. Then we have

23.17 $$-i\frac{\chi^3 g_{\alpha\beta}}{16\pi^4} \int_{\mathrm{M}} \frac{d^3\boldsymbol{p}}{2p_0\varepsilon} \operatorname*{Lim}_{\varepsilon \to 0^+} \int_{-\infty}^{\infty} d\tilde{p}_0 [\Theta(x_0^n - x_0^j) + \Theta(x_0^j - x_0^n)] \frac{e^{-i\tilde{p} \cdot (x^n - x^j)}}{\tilde{p}^2 + 2ip_0\varepsilon + \varepsilon^2}$$

For each node the Dirac current generates two propagators, one for the field and one for the adjoint. The field either annihilates a particle or creates an antiparticle, and is represented by an arrowed line pointing towards the vertex. The field $\psi_\alpha(x^n)$ at vertex n acting on vertex j, generates the propagator arrowed from j to n

23.18 $$\Theta(x_0^n - x_0^j)\langle\,\underline{x^n, \alpha}|\underline{x^j, \hat{\beta}}\rangle - \Theta(x_0^j - x_0^n)\langle\overline{x^j, \hat{\beta}}|\overline{x^n, \alpha}\rangle^{\mathrm{T}}$$

The Dirac adjoint field creates a particle or annihilates an antiparticle, and is represented by an arrowed line pointing away from the vertex. The adjoint $\hat{\psi}_\alpha(x^n)$ generates the propagator arrowed from n to j

23.19 $$\Theta(x_0^n - x_0^j)[\langle\overline{x^n, \hat{\alpha}}|\overline{x^j, \beta}\rangle] - \Theta(x_0^j - x_0^n)[\langle\underline{x^j, \beta}|\,\underline{x^n, \hat{\alpha}}\rangle^{\mathrm{T}}]$$

The time ordered product in 18.22 is unaffected under the interchange of (x^n, α) and (x^j, β). By interchanging (x^n, α) and (x^j, β) in the diagram, we find for the adjoint propagator arrowed from j to n

23.20 $$\Theta(x_0^j - x_0^n)[\langle\overline{x^j, \hat{\beta}}|\overline{x^n, \alpha}\rangle^{\mathrm{T}}] + \Theta(x_0^n - x_0^j)[\langle\,\underline{x^n, \alpha}|\underline{x^j, \hat{\beta}}\rangle]$$

23.20 is identical to 23.18, the expression for the Dirac propagator arrowed from j to n, so we do not distinguish whether an arrowed line in a diagram is generated by the field or the adjoint field. Similarly we find that the photon propagator, 23.15 is unchanged under interchange of the nodes, so we identify all diagrams which are the same apart from the ordering of the vertices and remove the overall factor $1/n!$ for a diagram with n vertices. By 21.13 and 21.14, 23.18 is

23.21 $$\Theta(x_0^n - x_0^j)\frac{\chi^3}{8\pi^3}\int_{\mathrm{M}}\frac{d^3\boldsymbol{p}}{2p_0}(ip\cdot\gamma + m)e^{-ip\cdot(x^n - x^j)}$$
$$+\ \Theta(x_0^j - x_0^n)\frac{\chi^3}{8\pi^3}\int_{\mathrm{M}}\frac{d^3\boldsymbol{p}}{2p_0}(ip\cdot\gamma - m)e^{ip\cdot(x^n - x^j)}$$

Use 23.3 and 23.4 in the first term, and use 23.5 and 23.6 and substitute $\boldsymbol{p} \to -\boldsymbol{p}$ in the second term. Then the propagator 23.21 is

23.22 $$-i\frac{\chi^3 g_{\alpha\beta}}{16\pi^4}\int_{\mathrm{M}}\frac{d^3\boldsymbol{p}}{2p_0}\lim_{\varepsilon\to 0^+}\int_{-\infty}^{\infty}d\tilde{p}_0[\Theta(x_0^n - x_0^j) + \Theta(x_0^j - x_0^n)]\frac{(ip\cdot\gamma + m)e^{-i\tilde{p}\cdot(x^n - x^j)}}{\tilde{p}^2 - m^2 + 2ip_0\varepsilon + \varepsilon^2}$$

Now collect all the exponential terms with $\boldsymbol{x}^n$ in the exponent under the sum 23.8, and observe that the sum over space is a momentum conserving delta function by 7.12. Then integrate over momentum space and impose conservation of momentum at each vertex, leaving an integral for each independent internal loop

23.23 $$\frac{\chi^3}{8\pi^3}\int_{\mathrm{M}}\frac{d^3\boldsymbol{p}}{2p_0}$$

Then only the time component remains in the exponents for the external lines 23.9 - 23.14. We introduce a finite cutoff $\Lambda \in \mathbb{N}$ by writing the improper integral

23.24 $$\frac{-i}{2\pi}\int_{-\infty}^{\infty}d\tilde{p}_0 = \lim_{\Lambda\to\infty}\int_{-\Lambda\pi}^{\Lambda\pi}d\tilde{p}_0$$

and instructing that the limits $\Lambda \to \infty$, $\varepsilon \to 0^+$ should be taken after calculation of all formulae. Then the photon propagator, 23.17 reduces to

23.25 $$-\frac{ig_{\alpha\beta}}{2\pi}\int_{-\Lambda\pi}^{\Lambda\pi}d\tilde{p}_0\frac{(1 - \delta_{x_0^n x_0^j})e^{i\tilde{p}_0(x_0^n - x_0^j)}}{\tilde{p}^2 + 2ip_0\varepsilon + \varepsilon^2}$$

For a Dirac particle, $p_0 > 0$, so we can also simplify the denominator under the limit $\varepsilon \to 0^+$. Thus the Dirac propagator arrowed from *j* to *n* is

23.26 $$\frac{-i}{2\pi}\int_{-\Lambda\pi}^{\Lambda\pi} d\tilde{p}_0 \frac{(1-\delta_{x_0^n x_0^j})(\tilde{p}\cdot\gamma + m)_{\alpha\beta} e^{-i(x_0^n - x_0^j)\tilde{p}_0}}{\tilde{p}^2 - m^2 + i\varepsilon}$$

The propagators, 23.25 and 23.26, vanish for $x_0^j = x_0^n$, and are finite otherwise, since the integrands oscillate and tend to zero as $p_0 \to \infty$. Loop integrals are proper and the denominators do not vanish so the ultraviolet divergence and the infrared catastrophe are absent, provided that the limits $\Lambda \to \infty$ and $\varepsilon \to 0^+$ are not taken prematurely (in the denominator of 23.5, ε^2 plays the role of a small photon mass commonly used to treat the infrared catastrophe). Thus the discrete theory modifies the standard rules for the propagators and gives heuristic justification to the subtraction of divergent quantities. But the propagators already incorporate the self interaction, and instead of renormalising we subtract a term which recognises that a particle cannot be annihilated at the instant of its creation. The standard rules are obtained by neglecting this term, and observing that for $\Lambda \in \mathbb{N}$, the sums over time in 18.22 act as $\tilde{p}_0$ conserving δ functions ($\tilde{p}_0$ is not energy, but it is equal to the energy of the measured, initial and final, states). Then renormalisation is interpreted as the removal of cutoff dependent terms arising from the second term.

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